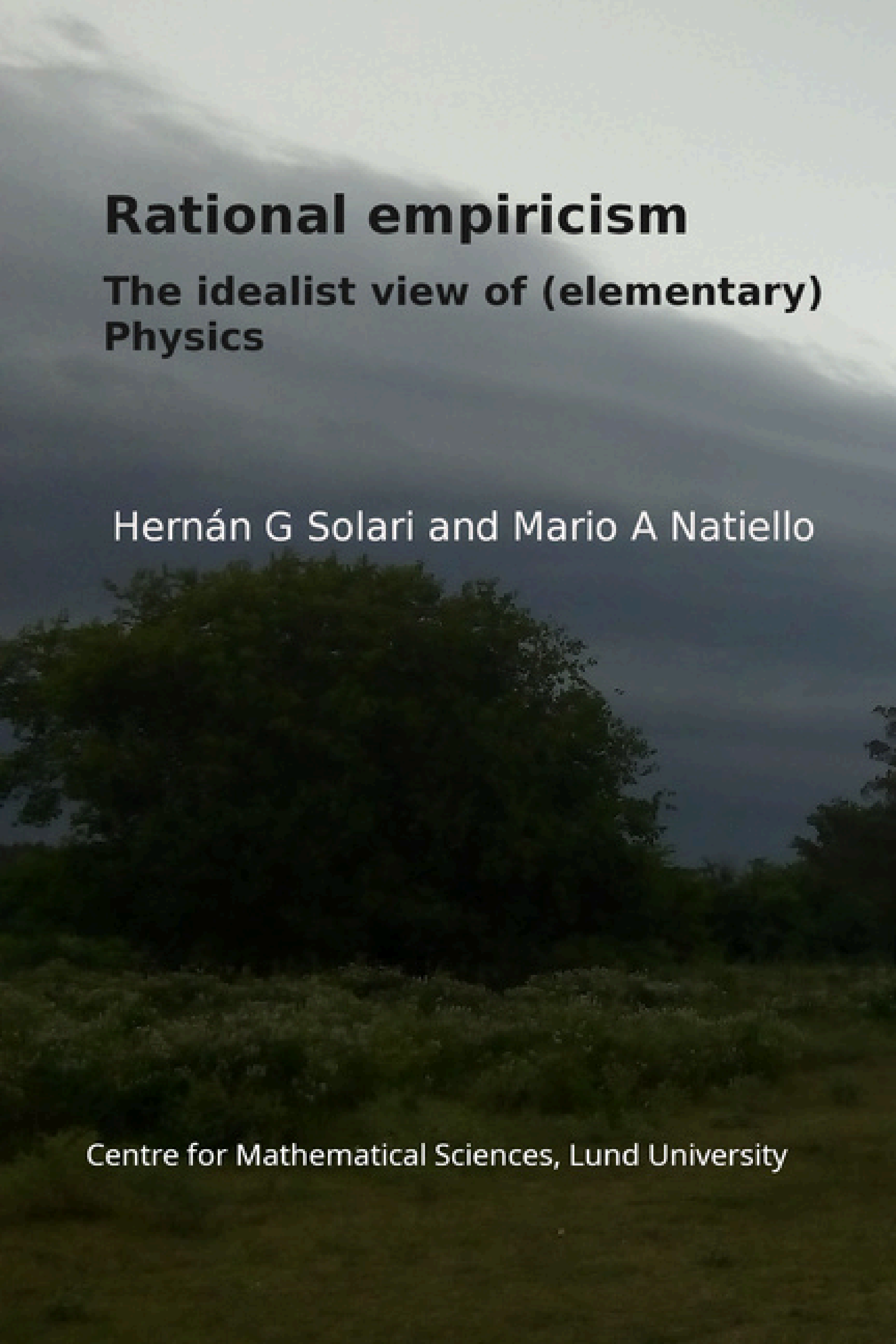

Rational empiricism

The idealist view of (elementary) Physics

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Rational empiricism. The idealist view of (elementary) physics.

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Front cover: A cold front advances covering the pampas with clouds. The clear skies are not longer at sight.

(Photo by F.I. Solari)

Back cover: Sooner or later the dry western wind known as "Pampero" restitutes the clear skies.

(Photo by F.I. Solari)

In the same form, clear thinking went out of sight with the second industrial revolution. Since then, we are waiting a "civilizatory pampero" that restitutes the reason obscured.

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Dedication

HGS: I dedicate this book to my daughter Florencia Ilusión and to all the living creatures that have been part of my life: Picaso, Botafogo, Shoot, Lianus and Valiente (horses); Bonzo, China, Chango, Reina, Princesa, Cachorrito, Leboni, Irina and Arwen (dogs); Lince, Cristal, Ladrillo, Bubu, Brillito, Borges, Luna, Zafira, Persefone, Sybil, Nara, Misha and Sir Terry (cats); the many trees in my life; and Bárbara. They taught me that it is not the "logic of domination" but rather the "logic of life" what allows us to understand the Universe and to be in harmony with it.

MAN: I dedicate this book to Patrizia (wife) and Lowe (child) who illuminate my life just by being there, and to the forests of Scandinavia and the varying landscape of Patagonia for letting me be part of them every now and then. I keep all of you with me.

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We also thank the different projects for digitising and collecting the works of the old masters. Without them, their writings would have been inaccessible to us. It is worth to mention the project Gutenberg, progetto Manuzio, The Internet archive, Anna's archive, Google books, the old Z-library and Lund University Library.

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Foreword. Rethinking physics.

> It is not possible to teach but it is possible to learn. As teachers we can only provide food for thoughts.

The aim of the present essay is to show in practice how what we call "scientific knowledge" depends on an internal determination that we brand *epistemic praxis*. If the itching of the doubt is the motor of thought and thinking seeks relief by achieving (at least temporarily) a belief, [CP 5.358, Peirce, 1994, The fixation of belief,] we must ask whether the conditions for the cessation of doubt and acceptance of a belief are the same for all producers of science[1]. We are not speaking here of the normal scientist [Kuhn, 1962] who has been trained at the University and had no other choice than accepting the authority criteria to progress in his career, a situation prolonged by the enforcement of the paradigm that the guild of scientists enacts in the form of peer's curation. We are interested in extraordinary science, the science that is produced when the phenomena, the observations and experimental results, manifest themselves in the form of a family of particular results that lack the unity provided by a theory. Then, a theory provides a coordination, a unity, for the uncoordinated collection of results that is as much as observations and experiments can provide, being them always singular (because of their temporal and spatial situation as well as the researchers involved to begin with).

The idea that knowledge is constructed contrasts with the popular idea of "knowing by experience". Most of those that claim to know by experience cannot explain how it works. It was Goethe who pondered about this matter. In his maxims style he wrote [Goethe, 1832]:

> 575. Everything factual is already theory: to understand this would be the greatest possible achievement. [...]
> 1071. When artists talk about nature, they always have the notion of an 'idea' at the back of their mind, without being clearly conscious of this fact.
> 1072. The same goes for all who stress the need for experience; they don't realise that experiential knowledge is only one half of experience.

Apparently, Goethe's teachings did not reach the empiricist minds. There seems to be a divide between minds. We read in Galileo Galilei:

> [...] io vi dico che quando uno non sa la verità da per sé, è impossibile che altri gliene faccia sapere; posso bene insegnarvi

[1] We will use the expression "producers of science" instead of the common "scientist" since in the past the production of scientific knowledge rested upon "natural philosophers". We will show later that the natural philosopher and the scientist differ precisely in their epistemic praxis.

> delle cose che non son né vere né false, ma le vere, cioè le necessarie, cioè quelle che è impossibile ad esser altrimenti, ogni mediocre discorso o le sa da sé o è impossibile che ei le sappia mai; e così so che crede anco il signor Salviati. E però vi dico che de i presenti problemi le ragioni son sapute da voi, ma forse non avvertite [p. 90, Galilei, 1997].[2]

According to Galileo recognising the Truth is something that is in us, that comes from the inside and cannot be taught.[3] For all those that as the product of self-consciousness (which in this case relates to philosophical intuition in [Husserl, 1983]) can relate Goethe's dictum to their own experience of constructing hypothetical explanations (theories), Goethe's maxims would result obvious. This kind is usually called *idealist*. In contrast, those that produce experience in a most natural way (called simple intuition in Husserl) identify themselves as *empiricists*. Thus, an idealist is an empiricist with self-awareness. For completeness, logical empiricists, under this perspective, belong to the empiricists community as they exclude from science the process of production of the axioms after which mathematical logic applies. We will latter examine how this happens from a historical perspective.

Truth is not to be sought in the words of the professor or the text of the book. Truth is written in the phenomena. To learn the Truth we must talk to the phenomenon, we must question it and become its friend. Only in recognition to our love the phenomena will speak out their Truth.

If not by the control that natural observations exert, theories could easily drift into metaphysics. Left under the commanding influence of authority, scientific knowledge degrades into religion (dogma).[4] Thus, reason and nature concur in the production of theories. In Galileo's words:

> Parmi, oltre a ciò, di scorgere nel Sarsi ferma credenza, che nel filosofare sia necessario appoggiarsi all'opinioni di qualche celebre autore, sì che la mente nostra, quando non si maritasse col discorso d'un altro, ne dovesse in tutto rimanere sterile ed infeconda; e forse stima che la filosofia sia un libro e una fantasia d'un uomo, come l'Iliade e l'Orlando furioso, libri ne'

[2][...] I say to you that if one does not know the truth by himself, it is impossible for anyone to make him know it. I can indeed point out things to you, things being neither true nor false; but as for the true — that is, the necessary; that which cannot possibly be otherwise — every man of ordinary intelligence either knows this by himself or it is impossible for him ever to know it. And I am sure that Salviati holds this opinion too. Therefore I tell you that the causes in the present problem are known to you, but are perhaps not recognized as such [p. 157, Galilei, 1953].

[3]You may ask for its genesis. The answer is: it comes from the *bildung* [Sorkin, 1983]. It is not possible to tell whether we all have the seed of self-consciousness or not. It just may happen that the seed died because of lack of nurturing or perhaps it is still dormant and waiting for development. What is certain is what Galileo taught, we must seek for it inside us.

[4]Normal science is then dangerously close to dogmatism as Popper realised [Normal Science and its Dangers, Lakatos, 1970].

> quali la meno importante cosa è che quello che vi è scritto sia vero. Signor Sarsi, la cosa non istà così. La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto. Ma posto pur anco, come al Sarsi pare, che l'intelletto nostro debba farsi mancipio dell'intelletto d'un altr'uomo (lascio stare ch'egli, facendo così tutti, e se stesso ancora, copiatori, loderà in sé quello che ha biasimato nel signor Mario), e che nelle contemplazioni de' moti celesti si debba aderire ad alcuno, io non veggo per qual ragione ei s'elegga Ticone, anteponendolo a Tolomeo e a Nicolò Copernico, de' quali due abbiamo i sistemi del mondo interi e con sommo artificio costrutti e condotti al fine; cosa ch'io non veggo che Ticone abbia fatta, se già al Sarsi non basta l'aver negati gli altri due e promessone un altro, se ben poi non esseguito. [Galilei, 1623][5]

Notice that this famous paragraph begins with a rejection of authoritarianism (scholasticism in this case). Galileo put the judgment by reason in scientific matters well above any other consideration, a characteristic that imprinted all the Enlightenment. For him reason is the instrument given by God to understand Nature, not human studies. Galileo writes:

> SALV. Se questo di che si disputa fusse qualche punto di legge o di altri studi umani, ne i quali non è né verità né falsità, si potrebbe confidare assai nella sottigliezza dell'ingegno e Bella prontezza del dire e nella maggior pratica ne gli scrittori, e sperare che quello che eccedesse in queste cose, fusse per far

[5] It seems to me, moreover, that I perceive in Sarsi a firm belief that in philosophising it is necessary to rely on the opinions of some famous author, so that our mind, if it does not marry itself with the discourse of another, must remain completely sterile and infertile; and perhaps he believes that philosophy is a book and a fantasy of a man, like the Iliad and the Orlando Furioso, books in which the least important thing is that what is written is true. Mr. Sarsi, this is not the case. Philosophy is written in this very great book that is continually open before our eyes (I say the universe), but it cannot be understood unless one first learns to understand the language and recognise the characters in which it is written. It is written in the language of mathematics, and the characters are triangles, circles and other geometric figures, without which means it is humanly impossible to understand a single word of it; without these it is a vain wandering in a dark labyrinth. But, even if we assume, as it seems to Sarsi, that our intellect must be made the slave of another man's intellect (I leave aside that he, by making everyone, and himself too, copiers, will praise in himself what he has censured in Signor Mario), and that in the contemplations of celestial motions one must adhere to someone, I do not see for what reason he chooses Tycho, placing him before Ptolemy and Nicholas Copernicus, of which two we have the entire systems of the world, constructed and brought to completion with the greatest skill; something I do not see Tycho having done, if it is not enough for Sarsi to have negated the other two and promised another, but then not carried out. (Unless otherwise stated, translations to English have been made by the authors with support by Google).

> apparire e giudicar la ragion sua superiore; ma nelle scienze naturali, le conclusioni delle quali son vere e necessarie né vi ha che far nulla l'arbitrio umano, bisogna guardarsi di non si porre alla difesa del falso, perché mille Demosteni e mille Aristoteli resterebbero a piede contro ad ogni mediocre ingegno che abbia auto ventura di apprendersi al vero. [pp.34-35, Galilei, 1997][6]

In summary, idealists and empiricists exert control of their theories in terms of agreement with observations (experiential input). In addition, idealists exert an additional control of rationality over the construction of theories. The questions are: is the control by experience enough? Could the rational control of the construction lead to a different theory? Is it possible that the demand of a rational construction were an excess that makes the construction impossible? The attempts at showing that rational control is mandatory have been, to our knowledge, consistently ignored by scientists. Let us give an example concerning the difficulty exposed by Galileo with regard to the Truth. Bertrand Russell [Russell, 1901] compares what he calls the "absolutiste" and "relativiste" theories of space:

> En ce qui concerne les nécessités de la pensée, la théorie kantienne semble amener ce résultat curieux, que tout ce qu'on ne peut s'empêcher de croire est faux. Dans le cas actuel, ce qu'on ne peut s'empêcher de croire, c'est quelque chose qui se rapporte à la nature de l'espace, non pas à celle de notre esprit. L'explication qu'on nous offre, c'est qu'il n'y a point d'espace hors de l'esprit; d'où j 'infère que toutes nos croyances inévitables au sujet de l'espace sont erronées.[7]

Russell concludes, opposing Leibniz and Kant (whom he attempts to ridicule), that the absolutist theory must be adopted. However, his argument fails to offer the required rational support. He assumes that space inevitably appears to our senses, but on the contrary, our senses can apprehend only spatial relations [p. 44, Solari and Natiello, 2023]. By ignoring the participation of our own mind, he can claim the existence of a space external to us.

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[6]SALV. If this dispute were about some point of law or other human studies, in which there is neither truth nor falsity, one could have much confidence in the subtlety of the mind and the readiness of speech and in the greater practice of writers, and hope that he who would exceed in these things would make his reason appear and be judged superior; but in the natural sciences, the conclusions of which are true and necessary and human will has nothing to do with them, one must be careful not to place oneself in defence of the false, because a thousand Demosthenes and a thousand Aristotle would be left standing against any mediocre mind that has the fortune of learning the truth.

An abridged version has spread in the WorldWideWeb: In questions of science, the authority of a thousand is not worth the humble reasoning of a single individual. Referred by Arago in "Laplace" [Notices Biographique, Tome Troisième, p. 513, Arago, 1865].

[7]As regards the necessities of thought, the Kantian theory seems to lead to this curious result, all that we cannot help believing is false. In the present case, what we cannot help believing is something which relates to the nature of space, not to that of our mind. The explanation offered to us is that there is no space outside the mind; from which I infer that all our inevitable beliefs about space are erroneous.

A science based on facts conceived as indubitable Truth (and then legitimate Axioms) as a result of simple intuition developed since the end of the XIXth Century. This notion of "fact" avoids the consideration on how to bridge the distance between what is perceived by the external senses and what is an idea in our mind.[8] In his discussion of absolute space, Russell comes to the point of saying “Du reste, on n’a qu’à pousser l’analyse un peu plus loin pour revenir à la région des simples faits; car nature de l’esprit reste encore un simple fait.”[9][p. 274, Russell, 1901]. This is somewhat surprising since from the time of Helmholtz foundational works in physiology it is known that the mind produces full ideas from very sparse stimuli. Ideas are not weak forms of sensations as Hume posed [10], they are “logicised” stimuli at the very least. The relation of ideas and perception was explored and exposed by Piaget who accounted on how the ideas of time and space are produced in our early life [Piaget, 1999]. Piaget can be recognised as ending the dispute using the experimental method: space and time are creations of the mind that the child uses to organise (logicise) the world. Piaget and García [1989] made clear the difference between observations and facts as well as the intervention of the subject in the production of facts. Kant was right in as much, for the adult, time and space appear as a priori intuitions, since no human remembers their life before the time when self-consciousness emerged and the *dialectical opening* creating the frame for posterior understanding was produced, resulting in: ego, universe/environment, space and time. The notion of dialectical opening was introduced in the construction of this work and will be explained later. Dialectics will be present throughout this essay. Dialectics usually emerge when what is perceivable is only the existence of differences but the elements that are perceived as

[8] The notion of fact is not free of ambiguity. A fact is what admits no doubt of being true (That the Moon revolves around the Earth is a fact), but then, what is a fact depends on our criteria for the cessation of doubt and acceptance (fixation) of belief. We do not observe the Moon rotating around the Earth, but what we do observe would be explained if the Moon rotated around Earth. This is enough for the “inference to the best explanation” [Lipton, 2004, McAuliffe, 2015] but it would certainly be not enough for Piaget-García or Peirce. Yet it is enough to put the conjecture under scrutiny. What else can be deducted, verifiable by observation, from what we already know (believe) and this new belief? If only after verifying the explanatory power of the new belief and its congruence with preexisting beliefs I admit it as a fact, then facts are the result of philosophical intuition, they are facts in Piaget-García or Peirce sense. Russell’s “all that we cannot help believing” introduces a category of matters that cannot be doubted, primary Truths, facts in Russell’s view. A science based upon facts, the axiomatic-deductive methods, theories as free invention are just some of the forms in which we prohibit doubting our inferences, a form to put a limit to the use of reason. If we further believe that the difference between impressions and ideas is just a matter of intensity as Hume did (see footnote 10), then we are entitled to perform “experiments” in our mind. The experiments in the mind cannot distinguish what is imagined (fantasy) and what is related to the observed, they simply propagate prejudices and demand us consistency with the adopted set of prejudices (axioms). Quite soon, correct axioms will have to be distinguished from incorrect but since the rational form of proceeding has been rejected we will have to pick a different form. The chosen method is described in [CP 5.382, Peirce, 1994] as a combination of “natural preferences” and social agreement. In the end, this method falls into *truth based upon authority*, the authority of the scientific guild in our case.

[9] Moreover, we have only to push the analysis a little further to return to the region of simple facts; for the nature of the mind is still a mere fact.

[10] All the perceptions of the human mind resolve themselves into two distinct kinds, which I shall call IMPRESSIONS and IDEAS. The difference betwixt these consists in the degrees of force and liveliness, with which they strike upon the mind, and make their way into our thought or consciousness [p. 7, Hume, 2011].

different cannot be isolated, do not preexist, hence being an opponent, a term in the differentiation (perhaps idealised as a limit form) which we cannot isolate from its opponent without destroying what is fundamental (essential) to the idea. To grasp the need for dialectics try to explain ideas like sweet, sour, acid, ... without making resource to pointing to something you can taste (sweet like in sugar, honey, ...) and not by contrasting with the other opponents listed.

The thesis of this essay is that Scientific theories depend on *epistemic praxis* and the latter depend on the *ethos of the time*. It might then happen that theories that we consider correct today can be shown at fault (to some degree) tomorrow without the need of new experimental results. We will show indicators suggesting that an abrupt change in the dominant epistemic praxis took place during the second part of the XIXth Century as part of the second industrial revolution.

We make an effort to recover the epistemic praxis of the Enlightenment period and to apply it to (the reconstruction of) Newton's mechanics (beyond the abridged textbook versions), disentangling as much as possible what comes from observation and what comes from reasoning. We also strive to be faithful to the originals and will indicate usual faulty matters in textbooks.

The lessons learned in constructing particular physical theories will be turned into lessons about constructing science. Both things are deeply entangled but we will only show a few steps in the dialectic.

Rethinking/reconstructing is not teaching. While the aim of university textbooks is to facilitate the use and acceptance of external ideas, rethinking aims towards constructing anew the theory as a rational sequence. When the educated reader thinks about mass they cannot use the concept of force since that concept requires the previous concept of mass. In teaching, didactic transpositions are permitted, but in constructing science they are not. In the end, the construction is an individual process.

Next we will tackle electromagnetism (EM) in the cautious version of Gauss and his followers (the Göttingen school) suppressing not only the material character of the ether but also its immaterial form[11] and completing Gauss EM with the ideas of Hendrick Lorentz and our own construction. The process results in a relational (instead of relativistic) EM which accounts for all observed EM features.

At this point of the development we will discuss how Enlightenment science degraded into normal science. And how the epistemological praxis was changed to make the latter acceptable ("correct").

Finally, a rival of current quantum mechanics is built from electromagnetism with the unexpected result of unifying all of classical physics.

The presentation of the different matters has not been completely straightened as it is usual in science although most of the dialectical production process has been suppressed; we have left some parts of the "becoming" to illustrate our use of reasoning. We very much regret that great minds, as for example Newton, have not left us indications on how they reached their conclusions and only communicated

[11] Using very little abstraction effort EM-fields considered as state of the ether at a given time or having their "site" in a geometric space-time are just different wordings for the idea of associating points in an assumed space to an electric and magnetic field.

the "last" argument and not those that despite being incorrect have taught us the process of rational supervision and the production of ideas. Those that want to make use of Newton's ideas have nothing to gain by following the becoming of them, yet those that would like to be Enlightened as Newton was would greatly benefit from an exposition of the becoming of ideas. True, there is no way to become like Newton by imitation, since after all, what we appreciate most of him is his originality. The most perfect copy will still be a copy, but listening to a master of thought may help us to find in ourselves our humble ability of independent thinking.

About the genesis of this essay. Loose ends and "paradoxes" occurring in Physics theories have puzzled us since the times of our undergraduate studies. Forcing students to accept contradictions is a form to force them not to rest on their reason but rather on authority. It promotes irrationality. As teachers we have made efforts to fill the gaps that often separate different matters making them mysterious, we have exposed false demonstrations in well established textbooks as well. Do we have to accept the Principle of Relativity because some thousand Einsteins tell us so? Are we not betraying Galileo by doing so? And if we repudiate Galileo, aren't we obliged to leave aside all that was constructed with his attitude until we reconstruct it with our irrationality? Can we find the rationale behind the Principle of Relativity? Does this rationale impose conditions on it? We anticipate that the answer is yes to both questions.

The expression "intellectual dishonesty" begins to hunt us. A famous scientist indoctrinated that theories are "free creations", however, he made every effort to persuade us with a mix of intuition, equations and deductions that his theories were correct and should be adopted. Nevertheless, there is nothing that can be considered correct or incorrect after admitting "free creation", this is, created by freeing us from the control of reason. If we repudiate the ether, shouldn't we repudiate any conclusion reached with its mediation unless we can produce it again avoiding unacceptable hypotheses? If your answer is yes, as we expect, then: why is it that we accept the Lorentz force? We will be told that we do so because of its transformations properties in front of Lorentz transformations. However, Lorentz transformations depend on constant relative velocities, they do not apply to accelerated situations. It cannot be decided by application of logic whether anything that follows from a false hypothesis is itself true or false. Free creation lifts us from the obligations that reasoning demands us to fulfil. Actually, how come that the relational motion between two apparatus which can be observed and measured becomes the motion with respect to the ether that cannot be measured? Which reasoning allows us to replace the motion relative to the ether by the motion with respect to a reference frame, one in a class of systems in which none of them can be approximated by real systems? In Galileo'sidealisation ideal relations are limit cases of actual relations. Thus, idealised systems are in the closure of actual systems and as such are in the limit of what is in the world. We can then agree that they are physical. In contrast, if the condition of being a limit is lifted, the ideal systems may very well be far away from experimental reach, they can be metaphysical or just fantasy.

What we see at the beginning of the XXth Century is desperation, the failure of hypotheses regarding the existence of the material ether might drop physics back some thirty years, and what is worse, it is a menace to the inner social structure of science. The leaders would have been forced to step down, their followers would have lost their privileged status, and the investment of the State in the developing of professional physics would had come into question. What happened then to reason? Was a fully rational approach to understanding Nature left behind for the benefit of scientific and socio-political leaders? When was reason abandoned? What has been left of reason? Which consequences has a relation with Nature not mediated by reason? Are we paying the price of our crime now? If we manage to survive the many forms that menace life on planet Earth, shouldn't we return to a relation of love with Nature? Shouldn't we return to a fully rational approach? In as much as Science has become a matter of social interest, science is a political issue and then, this essay is a political essay as well.

The abandonment of critical reason that characterised the second half of the XIXth Century continued during the early years of the XXth Century. Under the sole control of predictive success technoscience developed at a faster step than what it would be expected of a fully rational science. In as much as success at achieving our material goals becomes the highest value of the society, the new science was far superior to the old, rational, science. We were not surprised when around 2014, a claim (in scientific environments) about divergent series that can be summed up to a finite value became popular. A vague reference to "Hardy's book on series" circulated as support for the claim. A little research showed that Hardy didn't have anything to do with it. As far as we know, the book [Hardy, 1949] is a posthumous edition of Hardy's late lectures, produced after notes taken by his students, and it gave the clue to understand the inconsistency of the claim [Natiello and Solari, 2015].

Gradually, we incorporated other puzzling topics about how theories are constructed, how to monitor the construction, pinpoint errors, improve theories and how to succeed in enlarging our understanding. A strategy emerged: we had to learn how to reconstruct physics from a rational (idealist) standpoint beginning by classical mechanics where direct observations and intuitions make the work easier. Having learned this first lesson we would proceed to rebuild electromagnetism starting from the point in which the erroneous idea of the ether was incorporated. The construction of mechanics required a retrogress into the idea of space and time and in turn further retrogressing into some philosophical principles by Leibniz which after the development of group theory, and to some little extent category theory, were ready to be put in mathematical forms. The No Arbitrariness Principle lurks behind the Principle of Relativity as it lurks behind notions of Justice. The success with Newton's mechanics and electromagnetism made us to consider the principles of reason that were put to work in constructing theories. In this respect, the thoughts of Charles Peirce, William Whewell and Jean Piaget were enlightening. The same can be said of the references they made to Johann Wolfgang von Goethe and Georg Wilhelm Friedrich Hegel.

The last step in the construction of classical physics was the construction of quantum mechanics. Traditional quantum mechanics was constructed confessedly under an instrumentalist[12] epistemic praxis. Kragh [p. 80, 1990] reports in the scientific biography of Paul Dirac that he shared the belief of the Copenhagen-Göttingen camp about quantum mechanics being devoid of ontological content and that its value lay just in supplying a scheme that allows to calculate measurable quantities (cf. Subsection 3.4.1).

The quantum mechanics of textbooks is completely tainted by this perspective, although Dirac's honesty appears to have been forgotten. Constructing quantum mechanics under a rational perspective, a quantum mechanics with meaning, required to avoid following the traced steps. Our construction began from the conviction that electromagnetism and quantum theory are particular representatives of one theory obtained as a cognitive surpass[13]: of electromagnetism in which the radiation of the classically accelerated charges would be explained by the same theory that describes the atomic energy levels of the hydrogen atom, and where interaction of atomic entities with a measurement apparatus was straightforwardly described by electromagnetism. In the conception of Ludvig Lorenz, light and electromagnetic fields correspond to the form in which electromagnetic effects propagate at distance the consequences of an activity that occurs inside what we call matter and of which the only thing that can be measured are its effects. Matter is actually an inference, a clearly philosophical idea put forward

[12] Most physicists consider their science deals with the discovery of the laws of nature. However, during the early years of the XXth Century a most prominent group of physicists completely dropped this position and sustained that physics was merely a collection of useful formulae (see for example [Kragh, 1990]). Both groups of physicists would however agree that the relevant product of physics was the collection of formulae or plans which allowed them to make predictions. Actually, successful predictions, and insights, boosted technological applications justifying in front of the whole society the liberties granted to scientists. In short, predictive success was the key of the game called Science since late in the XIXth Century.

Under this philosophical perspective, called *instrumentalism* (a form of *utilitarianism*), the process of construction of theories –including their grounding– becomes less relevant and it is usually claimed to lie outside science (for example in [Popper, 1959]) or is substituted by a narrative that allows students to accept the desired formulae with lesser intellectual effort (sometimes called "didactic transposition").

[13] The concept of *cognitive surpass* [Piaget and García, 1989] is a guiding idea of the present work. It is a step in the development of our cognition by which what was previously viewed as diverse is recognised as a particular occurrence of a more general, abstract form. This cognitive step is dialectic in nature and displays the dialectic relation between particular and general. Cognitive surpass involves abstraction and/or generalisation, it does not suppress the preexisting theories but rather incorporates them as particular instances of a new, more general theory. It works in the direction of the unity of science[Cat, 2024], and as such its motion opposes the specialism characteristic of the instrumentalist epistemology. Cognitive surpass is a key ingredient of the abductive process since it is *ampliative* (i.e., it enlarges the cognitive basis) as opposed to the introduction of ad-hoc hypotheses to compensate for unexplained observations.

We notice that current physics hopes for a unified theory but little or no progress has been made in such direction. This matter should not come as a surprise since the requirements for understanding were downgraded to the level of analogy when adopting instrumentalism (see [Boltzmann, 1974, Mach, 2012]). Analogies hint us for surpass opportunities connecting facts at the same level, without producing the surpass. For example, quantum electrodynamics superimposes quantification to electromagnetism making mechanical analogies and populating an immaterial ether, the space, with immaterial analogous of classical entities (e.g., *springs*). What is preserved in this case is the analogy with classical material bodies, the recourse to imagination/fantasy.

by Faraday. When the mathematical advances made by Lorentz were merged in the Faraday-Gauss-Lorenz perspective of electromagnetism, it became clear that charge densities and current densities were phenomenological proxies for the electrical activity inside matter as they emerge, mathematically speaking, as the Lagrange multipliers associated to Lorenz idea. Hence, what was needed to complete the theory of matter along Faraday's thoughts was a constitutive model of matter which would be coupled to Maxwell's fields in the form proposed by Lorentz. It was then time to look for the seeds in de Broglie and Schrödinger since what came from the Copenhagen-Göttingen camp was confessedly meaningless. The work by de Broglie became soon marred by his trust in special relativity as the keystone of electromagnetism. Actually, in the famous 5th Solvay Conference of 1927 from which Bohr's statistical interpretation of quantum mechanics emerged as the most popular theory, there were only two participants (among the most famous) that were in dissidence: Einstein and Schrödinger. Yet, all the participants were supporters of special relativity. The ether definitively faded out around 1930.[14]

At the time of the conference the choices were: a. stay with special relativity; b. move back to the ether (20 years regression); or c. go back to the age before the introduction of the ether (about 60 years regression). Unless one is equipped with some new insight or tool that our masters did not have at hand, the choices b. and c. sound insane. Special relativity had no rival since Gauss' ideas had previously been suppressed by the social pressure of the adherents to the ether. An attempt to reconsider Gauss' ideas made by Carl Neumann at the beginning of the XXth Century was simply ignored and let to die by the physics community [Jungnickel and McCormmach, 2017]. The attempts of reconciliation of electromagnetism and quantum theory on the basis of field theories made by Dirac and Feynman led to the mathematical problems commented above. In fact, the theory was called back by its creators. A mature Dirac said:

> Some physicists may be happy to have a set of working rules leading to results in agreement with observation. They may think that this is the goal of physics. But it is not enough. One wants to understand how Nature works." [...] "But the price one must pay for this success is to abandon logical deduction and replace it by working rules. This is a very heavy price and no physicist should be content to pay it.[pp.184-185, Kragh, 1990][Concerning successful agreements in *renormalisation theory*]

And similar words by Feynman,

> The shell game that we play [...] is technically called 'renormalization'. But no matter how clever the word, it is still what I would call a dippy process! Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent.

[14] Some of the latest works based upon the ether were published by Michelson [Michelson, 1925, Michelson and Gale, 1925].

> It's surprising that the theory still hasn't been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate.[Feynman, 1983]

Once a theory has been adopted by a community and each member has invested time and effort at learning and mastering it, once the scientist is no longer a human being working in QED by rather a quantum relativity expert (i.e., what he does is the primary determination of who he is) calling back the theory represents taking away the working social capital of the community. While Feynman and Dirac were socially wealthy and could afford the loss, the poor followers could not do it and they did not. A conflict of interests developed, the private interest of the community guarding the paradigm and the public interest of humanity: Truth. Once again, the private interest prevailed. Irrationality further advances.

Einstein's objections to quantum mechanics are well known [Einstein et al., 1935]. Less known are the objections by Schrödinger who pointed out that there is no support for the pointlike particle hypothesis and that adopting it calls immediately for the statistical interpretation of quantum mechanics [Schrödinger, 1995]. Schrödinger objections met the same attitude that Neumann's, they were ignored by the community. Not refuted, just ignored. The same fate was suffered by others, it can be said that it is the standard social reaction of the physicists community to sound challenges. We expect this work to elicit such reaction as well. Smolin and Harnad [Smolin and Harnad, 2008], as well as Woit [Woit, 2011] met recently this social treatment when they objected string theory; Essen [Essen, 1971, 1978], Dingle [Dingle, 1972] and Phipps [Phipps Jr, 2006, Phipps, 2014] were attacked and ignored when they raised solid objections to special relativity. A long list of applied physicists and engineers must be added, noticeably experimental works by Müller [Müller, 2014], Kelly [Kelly, 2004] and others that expose the failure of Faraday's law as stated in textbooks (adapted to the field concept) a matter sorted out in [Munley, 2004]: Faraday's original law does not fail. It is the change of meaning of the symbols operated to satisfy first the ether fantasy and next field theories what results in a straw version of the theory that fails in some circuits with moving parts. The extensive work by Moon and coworkers [Moon and Spencer, 1954, 1956, 1960, Moon et al., 1991, 1989a,b] has been easily ignored belonging the authors to the engineers' community. Even some observations that appear to challenge current beliefs [Bilbao, 2016] are consistently ignored.

In conclusion, it is not enough to show that a theory is wrong and it has been refuted, it is necessary to offer a substitute at the lowest possible social cost for the scientific community if the old theory is going to be dropped. We are persuaded that the private social interest will only yield to reason when confronting a catastrophic failure. We can only provide a contribution to the restoration of reason, the catastrophe appears to be in the making. The epistemic praxis that has led to the current planetary crisis, the utilitarianism that we utterly object, is the main responsible for the rapid deterioration of the conditions for life in the Earth.

A rational construction of quantum mechanics is possible, it explains every experimental result available, it is consistent with electromagnetism and classical mechanics by construction, requires no interpretative discourses and, as a gift, it unifies in one formulation classical mechanics, electromagnetism and quantum mechanics. This last step in our essay shows as well the relevance of the concept of "consilience" introduced by Whewell. Peirce's principle of reality, consilience, non arbitrariness and Piaget-García's cognitive surpass are the constructive tools of Science we have at hand.

Contents

CHAPTER 1

The construction of mechanics

The contents in this Section are based in previous work developed in [Solari and Natiello, 2018, 2021], on which we shall presently elaborate.

In order to construct a rational theory of mechanics it is necessary to develop the concepts of *space* and *time*. However, this is not the actual starting point. Developing concepts requires an active exercise of our mind, which we have been training since the moment of birth. We permanently perceive stimuli, along with an urge for organising them. The construction process for space, time and other concepts in our early childhood was studied by Piaget [1999] through extensive experimentation. We read,

> To understand how the budding intelligence constructs the external world, we must first ask whether the child, in its first months of life, conceives and perceives things as we do, as objects that have substance, that are permanent and of constant dimensions. If this is not the case, it is then necessary to explain how the idea of an object (object concept) is built up. The problem is closely connected with that of space. A world without objects would not present the character of spatial homogeneity and of coherence in displacements that marks our universe. Inversely, the absence of "groups" in the changes of position would be equivalent to endless transformations, that is, continuous changes of states in the absence of any permanent object. [p. 3, Piaget, 1999]
>
> The conclusion to which the analysis of object concept has led us is that in the course of his first twelve to eighteen months the child proceeds from a sort of initial practical solipsism to the construction of a universe which includes himself as an element. At first the object is nothing more, in effect, than the sensory image at the disposal of acts; it merely extends the activity of the subject and, without being conceived as created by the action itself (since the subject knows nothing of himself at this level of his perception of the world), it is only felt and perceived as linked with the most immediate and subjective data of sensorimotor activity. During the first months

> the object does not, therefore, exist apart from the action, and the action alone confers upon it the quality of constancy. At the other extreme, on the contrary, the object is envisaged as a permanent substance independent of the activity of the self, which the action rediscovers provided it submits to certain external laws. Furthermore, the subject no longer occupies the center of the world, a center all the more limited because the child is unaware of this perspective; he places himself as an object among other objects and so becomes an integral part of the universe he has constructed by freeing himself of his personal perspective. [p. 97, Piaget, 1999]

The newborn child gradually recognises a separation with respect to the environment. By exercising/developing our memory we perceive some degree of permanence of ourselves, *ego*, and thus an idea of identity. Together with ego, we recognise that there exists something that is not myself: *not-ego*, (the environment). Memory shapes identities in not-ego as well through attributes that appear to return upon every observation. Thus, through the same process the child becomes self-aware and aware of the real world, which is populated by identifiable entities. We no longer experience the world anew at every eye-blink, but recognise a sort of permanence in change.

If we want to recognise permanence we have to begin explaining that there was "then" and there is "now", that "then" is not "now" which prompt us to say: something has changed. Since the time of Aristotle, change in its pure form —in essence— is called time. What appears as permanent in the comparison between what is perceived and what is recalled from (some form of) memory is what we name substance, the thing. If the child observes a scene of the playground they may observe that it is not the same it was then, perhaps now they must make more effort than then to reach a toy from the cradle. If not for this small inconvenient, the toy would be the same that it was then, there is something permanent associated to the toy, and yet, something has changed. If the child focuses attention in the toy itself, and manages to ignore the rest of the signals reaching them, they will recognise that the toy is their toy, it has not changed, it has an identity (that what is unchangeable in it, permanent) and circumstance or state (that what changes in it) as its relative position (perhaps measured as difficulty of reaching: close or far away?) with respect to other things as for example the cradle. We cannot reach the concept of "thing" without some perception of time and some perception of circumstances including spatial relations. The perception of one of them requires some implicit perception of the other (e.g., we need permanence to perceive change and conversely). Space, time, thing, ego are primitives of explanations; we cannot explain them from other previous concepts. For the adult, they are a priori intuitions as Kant put it, but they are constructions of the child in his adaptation to independent life in the Universe. We call this general situation a *dialectical opening*. A new set of interrelated ideas is born out of the opening and these relations constitute a new vision/organisation of the world, a mental opening.

The exploration continues, the toy can be placed at different "places", i.e., its spatial circumstances depend on time (can change). Children soon learn that the toy that is out of sight has not ceased to exist and does not come into being every time it comes into sight. A simpler explanation is that it remains being the same toy despite not being at sight. It is "better" because it explains that all the characteristics the child has in memory are still there in the new apparition of the toy. Our child is constructing theories and picks them to minimise the surprise, they are building regularity in the world, making a cosmos out of the chaos. *Abduction* becomes present as an innate constructive mechanism. As soon as they experience the failure of their conjectures the child will learn forms of perfecting them, to put them to test. Moving things back and forth, changing their place (their relation with other things) is going to be a frequent activity. With this activity the child learns that translations of the toy can be combined to produce a new and different translation; doubtfully they will introduce the null displacement since the idea of zero is a difficult one, but they will easily conceive that translations can be undone bringing the toy back to the previous place. Although they are not conscious of it and much less of the abstract form, they associate to translations the structure of a group (for a better explanation in full detail see [Chapter 2, Piaget, 1999]).

In turn, time is a different matter. Even if they manage to reproduce the same disposition of things in the room that is in their memory children realise that time was not turned back, that there are things not under their control. Perhaps the child was tired after the effort, which means that the child's state has changed, perhaps it was something in the environment. Unlike the spatial relations in their small universe that can be reverted, time cannot be reverted, it belongs to a sphere of things out of their control. There is then a "system" (using the vocabulary of physics) which is the subject for the child's intervention and curiosity and the environment (the complement of the system in the Universe). The system appears as under control, the environment as autonomous although this separation is ideal, i.e., the result of another mental operation, a limiting case as Galileo taught us. The child will learn more advanced notions of time later[15]. Time is constructed in the mind as the result of temporal sequences, even the notion of speed can precede the notion of time, since the winner of a running race over a fixed distance is easily associated to the faster runner without the need to consider time. Thus, anything in motion makes a clock, any perceived change can be used to make a clock. There is nothing special to it, the day, the lunar month, the sun year impress us as being at our easy disposition, but any motion, repetitive like the pendulum or not, can make a clock. However, relative positions belong to the system while time belongs to the environment or the Universe. Positions are under our control and have associated an operational group while time belongs to the Universe and there is no operational group associated to it. The classroom experience of plotting time as if it were space can bring a good amount of confusion to the instructed reader.

[15] Teaching how to read the clock (old analogical clock) had usually to wait until the children had matured their sense of time, perhaps around 10 years of age.

There is more than distance in spatial relations. Sooner or later we make plans and give suggestions, for example on how to reach fruit from a tree. We consider the case when the spatial relations between all the objects are kept unchanged while we move, i.e., we just change our position in the scene. There are multiple forms of giving the instructions following different paths, but all of the paths when considered from beginning to end would allow us to reach the fruit in the same form. Repeated experiences at grabbing fruits and reaching toys teach us that instructions (or plans) require three independent spatial references we call (local) directions. The space of paths must be such that it reflects the operational group discovered as children at an early age. The composition laws discovered as children playing with objects, the possibility of adding useless closed loops to our path to reach the fruit, the existence of a path to come back once we have grabbed the fruit are now properties of the space we have constructed. We infer that its (local) dimension is three and no evidence has appeared so far (in terms of the life of humanity) to think otherwise. We are certain that we have some control of spatial relations and no control of temporal relations, the clock is outside the system. The usual problems with the concept of infinity pop up here. Can we consider the Universe as an outside system? We ought to understand that the decomposition of system+environment is present in all of our experiences, it is a constant and no matter how we consider a sequence of ever larger systems, the limit of a constant is itself. The laws of physics apply not to the Universe but rather to systems immersed in the Universe.

The process of developing during the first two years of life is described as:

> Corresponding to this process of evolution related to intellectual behavior is a sort of law of development of knowledge, as we have seen through the analysis of the concept of object and of space. The initial state is that of a universe which is neither substantial nor extended in depth, whose entirely practical permanence and spatiality remain related to a subject ignorant of himself and perceiving reality only through his own activity. The final stage is, on the contrary, that of a solid and vast world obeying physical laws of conservation (objects) and kinematic ones (groups), in which the subject places himself consciously as an element. From egocentrism to objective relativism seems to be the formula of this law of evolution.[p. 219, Piaget, 1999]

According to Piaget during this age period there is a development of the notion of causality. What is meant as causality deserves explanation. Piaget does not refer to an explanation of how things happen, the child is seeking no explanation but rather to make their actions manipulating the world "at hand" more efficient. What Piaget calls "causality" has been called by Peirce "belief". For C. Peirce [CP 5.371, Peirce, 1994], a belief is what conducts our actions and not a explanation nor any kind of declaration[16]. The child soon learns about their own actions as causes and gradually identifies other causes and relations that seem to be independent of the child's own actions. Causes always precede effects or occur at the same time with them. When they seem to be simultaneous it is reasoning what determines which is the cause and which one is the effect[17]. Older children will put to test received beliefs[18]. The process of adaptation to the world involves then the construction of theories that allow us to predict the outcome of some events (effects) with some anticipation when we know the causes. The search for the causes is a regression, it can be called abduction or retroduction [CP 6.98, Peirce, 1994] and seeks to establish a belief, it is reasoning in a direction opposite to deduction. Deduction goes from causes to effects following an established theory. Abduction infers causes from effects, constructing theories.

The natural philosopher strives to develop laws for the understanding of Nature by pursuing and deepening the initial effort of the children seeking trustworthy relations (theories) that can optimise their own actions. Understanding the relation ego-environment enhances our survival chances. The question arises: how do natural laws come about? Are they intuitions of powerful minds which we accept inasmuch they are useful pragmatic beliefs? If yes, what do we do when these beliefs fail and prove to be anything but useful? Is there room for critical reason, as in

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[16] In terms of justice, the "doctrine of own acts" is based on the same perception, what we really believe manifests and guides our acts.

[17] Kant explained this as follows [pp. 183-184, Kant, 1787]:

> For example, there is heat in a room, which does not exist in the open air. I look about for the cause, and find it to be the fire. Now the fire as the cause is simultaneous with its effect, the heat of the room. In this case, then, there is no succession between cause and effect as regards time, they are simultaneous. Still, the law holds good. The greater part of operating causes in nature are simultaneous with their effects, and the succession in time of the latter is produced only because the cause cannot achieve the total of its effect in one moment. But at the moment when the effect first arises, it is always simultaneous with the causality of its cause, because, if the cause had but a moment before ceased to be, the effect could not have arisen. Here it must be specially remembered that we must consider the order of time and not the lapse thereof. The relation remains, even though no time has elapsed. The time between the causality of the cause and its immediate effect may entirely vanish, and the cause and effect be thus simultaneous, but the relation of the one to the other remains always determinable according to time. If, for example, I consider a leaden ball, which lies upon a cushion and makes a hollow in it, as a cause, then it is simultaneous with the effect. But I distinguish the two through the relation of time of the dynamical connection of both. For if I lay the ball upon the cushion, then the hollow follows upon the before smooth surface; but supposing the cushion has, from some cause or another, a hollow, there does not thereupon follow a leaden ball.

[18] A personal story illustrates this point. Grandma is playing with her granddaughter, who is about three years old and sits in a tall chair designed for children. The child tries to reach a glass that sits on the table. Grandma warns her of the dangers involved and how the glass will break if it falls from the table onto the floor. The child turns her hand and pushes slowly the glass towards the border of the table with the dorsal side of her hand. Grandma understands what she intends to do, the child makes visual contact checking for a disapproving sign but no one comes. She gives the final push, the glass falls and breaks in a hundred parts. The child smiles. She would never again let a glass fall. The theory appears to be correct. The child learned something else, more important: grandma is trustworthy.

Hegel's dictum: "progress in philosophy is rather a retrogression and a grounding or establishing by means of which we first obtain the result that what we began with is not something merely arbitrarily assumed but is in fact the truth, and also the primary truth." [§ 101, Hegel, 2001a]?

One of the simplest forms of theoretical thought is what is usually called experience. Experience is a form of causal relation in which we come to expect phenomena on the basis of the observation of previous phenomena alike. This description however hides more than it reveals. It hides how previous observations were gathered together and what is meant by being alike. It hides that only the singular and unique can be a matter of observation and there is a mental process in which the singular becomes a particular of a general (abstract) phenomena. It further hides that the gathering together was produced in consideration on what it wants to explain, this is, the abstract class was constructed by its explanatory character, it did not preexist it and each observed phenomena might very well belong to different abstractions depending on what we are trying to explain in each situation. It hides that what we conceive as belonging to opposite classes (dichotomies) may very well belong in the same class at a higher abstraction level, hence that there is equality in the difference. The empiricists that declare that their knowledge is pure experience merely reveal their lack of consciousness (and then of control) of the process of producing experience. It was Goethe who wrote extensively about experience [Goethe, 1832].

In textbook mechanics, what actually is the result of reason is often presented as resulting from experience. For example, that the classical addition of velocities[19] is the result of experience alone is at the very least strange. If it corresponds to not reported measurements it should at least be reported with measurement error or as $w \sim (v + u)$ since the exact equality does not apply. If what is implied by experience is experience in the successful use of the expression to obtain other relations that somehow matched observations we still cannot eliminate the observational imprecision and perhaps we would better speak of habits rather than experience. In any case, how do we get the equal sign? We will show that it is the result of idealisations and the use of a principle of reason. It is reasoning what has the required infinite precision. If the law were to be considered as just a summary of experimental results and enunciated as $w \sim (v + u)$ it should be doubted in regions not explored by the experimental support of it. In contrast, if it results as a requirement of reason, it should be demanded that any replacement satisfies those requirements. However, the latter requires consciousness on how we arrived to the law and this consciousness is what distinguishes the idealist from the empiricist. The empiricist wants to close the discussion, wants the doubting to cease in the statement "I know it by experience" as Russell did (see footnote 7).

[19] It states that if the mobile A moves with velocity v relative to the mobile B, and the latter moves with velocity u with respect to a third mobile C, then the velocity w of A with respect to C is given by $w = v + u$.

1.1. Mathematisation of the description

The relation between mathematics and reality is dialectical. As clearly discussed by Engels [p. 23, Engels, 1878] mathematics emerges from our efforts to organise the stimuli of our interaction with Nature.

> [...] it is not at all true that in pure mathematics the mind deals only with its own creations and imaginations. The concepts of number and figure have not been derived from any source other than the world of reality. The ten fingers on which men learnt to count, that is, to perform the first arithmetical operation, are anything but a free creation of the mind. Counting requires not only objects that can be counted, but also the ability to exclude all properties of the objects considered except their number – and this ability is the product of a long historical development based on experience. Like the idea of number, so the idea of figure is borrowed exclusively from the external world, and does not arise in the mind out of pure thought. There must have been things which had shape and whose shapes were compared before anyone could arrive at the idea of figure. Pure mathematics deals with the space forms and quantity relations of the real world – that is, with material which is very real indeed. The fact that this material appears in an extremely abstract form can only superficially conceal its origin from the external world. But in order to make it possible to investigate these forms and relations in their pure state, it is necessary to separate them entirely from their content, to put the content aside as irrelevant; thus we get points without dimensions, lines without breadth and thickness, a and b and x and y, constants and variables; and only at the very end do we reach the free creations and imaginations of the mind itself, that is to say, imaginary magnitudes.

However, in the same form, it can be said that mathematical reasoning structures and modifies our perception of nature. Mathematicians, as Euler, excelled at writing in mathematical terms what came from our observation of the world. Thus, observations by the senses produce ideas and these ideas can be presented in mathematical language, which may need to be created to the effect. Mathematical/logical operation on the mathematical idealisation leads in time to unsuspected results that were written in our beliefs, although we had not developed any awareness of them. These new ideas resulting from mathematics can be put in relation to the observed again by intuition (in this case interpretation). We will explain the details of this process in Chapter 2. We call mathematisation to the process that goes from the observation up to the construction of mathematical relations representing it, including the necessary checks that the construction faithfully reflects the relations whose intuition made it possible. For example, if we consider space, its mathematical form must reflect what we have learned from space. Failing in satisfying this requirement, such form is just pure invention, fantasy, and no claim about reality can be made through it.

If we want to know about the *physis* we have to decontaminate any observation from the contribution of the observer since otherwise we only express subjective opinions about the phenomena. This requirement will in time be formulated as mathematical conditions on mathematical expressions.

Our first belief is:

PRINCIPLE 1. *There is a material world we perceive with our senses (including experiments).*

This principle cannot be proved, but if it were true and we deny it, we cut ourselves away from any possible rational description. The word "material" in it deserves some attention. Most of us would consider matter something that can be inspected by our senses. This is the original intuition. The principle then says that we acknowledge that there are signals that reach our external senses, signals that do not originate in ourselves, hence we conjecture that there is an origin of those signals and we call it "matter". What is perceived by the senses are the signals, matter is a mental construction, a belief. Matter is the cause of our perceptions, it is inferred and as such a little bit less real than the signals which impact our senses. This point of view was put forward by Faraday

> A mind just entering on the subject may consider it difficult to think of the powers of matter independent of a separate something to be called the matter, but it is certainly far more difficult, and indeed impossible, to think of or imagine that matter independent of the powers. Now the powers we know and recognize in every phenomenon of the creation, the abstract matter in none; why then assume the existence of that of which we are ignorant, which we cannot conceive, and for which there is no philosophical necessity? [p. 291, Faraday, 1844]

The identity of matter is then to act upon us and to be acted upon by us. Matter cannot be separated from its active and passive determination. If we consider that anything in the material world is determined by this possibility of acting and be acted on, then we cannot avoid recognising that we are in the material world. Together with the material world there is a second world, the word of ideas. As far as we know, ideas require a material support to live and they are not associated with intensification of interactions. Some of our ideas are directly linked to Nature while others are not, they have invented parts, elements that do not have a correspondence with observations. The notion of the two worlds is an ancient one and was presented by Plato in several of his writings, for example [Plato, 2014]. For the ancient Greek, our material world was a world made of copies in clay. The real world was the world of ideas, the material world was made of (bad) copies of the ideal forms. It was Galileo and his mates (think for example of Johannes Kepler) who turned the relation upside down; the world of ideas constitutes an idealisation of the material world. Idealisation is a limit procedure, we expect to find material systems that are good examples of our ideas about Nature, but perhaps not the exact realisation of them. Ideas about Nature produced by idealisation are representative of material situations that lie in what can be called a "neighbourhood" of the idea. Thus, the idea is an abstraction and the elements in its "neighbourhood" are particular material systems that differ in one way or another from the idea. What we call laws of physics are synthetic relations of concepts/ideas associated to idealisations. Picking one material representative for the idea and not any other of those available implies an arbitrary decision. It is arbitrary because it is not determined by the idea but rather by our picking. We then postulate a principle of knowledge:

PRINCIPLE 2. **[No Arbitrariness Principle (NAP)]** *No knowledge of Nature depends on arbitrary decisions.*

Briefly, *if there is no reason for it, we shall make no difference. However, if for the sake of the presentation we find convenient to make differences, the different presentations will be related by the group of arbitrariness.* In this form it relates to Leibniz' principle of sufficient reason [Ballard, 1960].

The rejection of arbitrariness corresponds to the acceptance of reasonability since arbitrary is the word for those things not based on reason. Thus, NAP corresponds to the demand for the laws of Nature to be rational. Our construction can be called rational empiricism and contrasts logical empiricism in as much as logic is only a part of reason and it is subordinated to reason. [CP 2.191, Peirce, 1994].

1.1.1. Objectivity. Spatial descriptions need not be the same for all observers but just corresponding ones. Let x_{CB} be a symbol for the idea of relative position between e.g., a cedar tree and a birch. We are used to think that the observer with reference a represents it as $x_{CB} = x^a_C \ominus x^a_B$, which is in fact a synthetic judgment—not a mathematical one— since it proposes a relation between three different concepts: the positions x^a_C, x^a_B of cedar and birch relative to the observer a and the relative position between the trees, x_{CB}. The operation $\ominus$ may be worded as: Given the instructions to go from a to C previously passing by B, x_{CB} corresponds to the final portion of instructions, after B is reached. It is important to realise that x_{CB} is a symbol that relates to the sensory perceived world, not an internal mathematical object.

The observer entered the picture by the arbitrary action of referring positions with respect to a. To attain objectivity and comply with the No Arbitrariness Principle it is required to remove the subject. Hence, while x^a_C, x^a_B include the subjectivity of a, reason demands x_{CB} to be independent of the choices of the observer (a), since the first principle postulates that reality is independent of the observer. We say that relative position is *objective*, despite x^a_C, x^a_B being *subjective* since they depend on the subject. x_{AB} can be said as well to be *intersubjective* since it is conceived in the same form by all observers. We presently formalise the mathematical expression of objectivity.

Consider a set E of concepts/magnitudes and a set A of arbitrary decisions, necessary for its representation. Let $R_a(e)$ be a representation of the concept $e \in E$ depending on the arbitrary decision $a \in A$ and let F be a "natural law" expressed as $F(R_a(e)) = 0$. For example, if the concept e is "relative distance" (between objects at relative rest), and a is a choice of origin for Cartesian space, $R_a(e)$ is the real number giving the distance according to a and the "natural law" is some mathematical expression involving this number, e.g., $R_a(e) - C = 0$, for the "natural law" expressing that this distance has the constant value C.

Definition 1.1. (Objectivity) A natural law is objective if $F(R_a(e)) = 0$ holds for any $a \in A$.

Consider an invertible transformation T_{ba} that maps a representation R_a onto another representation $R_b = T_{ba} R_a$. Definition 1.1 can be restated as:

$$F(R_a(e)) = 0 \Leftrightarrow F(T_{ba} R_a(e)) = F(R_b(e)) = 0.$$

NAP demands that the transformations T_{ba} form a group. Indeed,

(1) $T_{ab} \circ T_{bc} = T_{ac}$, there exists a composition law.
(2) $T_{aa} = Id$, there exists an identity.
(3) $T_{ab} = (T_{ba})^{-1}$or $(T_{ba} \circ T_{ab}) = Id$, an inverse exists.
(4) $T_{ab} \circ (T_{bc} \circ T_{cd}) = (T_{ab} \circ T_{bc}) \circ T_{cd}$, the composition law is associative.

We call this group the *arbitrariness group*. Objectivity is the core in F that persists after modding out arbitrariness.

1.1.2. Space. From the initial intuition of recognising entities at a relative distance of ego we may construct an organising structure. Consider first objects that are at rest with respect to ego, say plush toys representing animals spread in the room. Soon we will notice that such objects are also at rest relative to each other. We ask the child "go to the dolphin" (or any other) and the child knows the steps required to grab the dolphin. The child can also go from the dolphin to the donkey, etc. The mathematisation of this game suggests that to reach an object we produce instructions by concatenating steps in different directions. The instructions to attain a given target are not unique. Along each direction there is an ordering: smaller number of steps require less effort than a larger amount. More experimentation indicate that all instructions can be described as steps along no more than three independent directions, that the concatenation of two instructions is itself also an instruction, that there is a "return" instruction that takes us from the target to our starting position, a "do nothing" instruction that leaves us wherever we are, and eventually that the instructions can be taken in any order (they are commutative). Cartesian space, namely the euclidean space $\mathbb{R}^3$ with a distinguished point called origin is the mathematical structure spawned from these ideas.

While the construction may reflect the child's initial view of ego, it still has the arbitrariness of the choice of origin and of the orientation of the three directions. Every "ego" has a Cartesian space, but relative position between two objects at relative rest cannot depend on the choice of origin/orientation.

Repeated experiences convince us that two entities A and B that are at rest relative to ego, are also at rest relative to each other. In terms of connecting instructions, the instructions to go from A to B are always the same (including the direction, which is universally defined by A and B) and they do not depend on the choice of reference or the location of other objects. Moreover, the instructions for going from A to B and from B to A are inverse of each other. It is a matter of retracing our steps in the opposite direction. This property is inbuilt in Cartesian space. While the vector connecting the reference point (origin) and each of A and B may change when we change the reference point, the relative position between two objects at relative rest has to be independent of the particular choice of Cartesian space.

AXIOM 1. *A reference point (origin a) and a set of three independent directions in the real (sensorial) space is selected and represented by the symbols $\hat{\mathbf{e}}_i$. The position of an object O is then represented by $x^a_O = \sum_i x^a_{Oi}\hat{\mathbf{e}}_i$, with x^a_{Oi} real numbers.*

There are a couple of hidden assumptions in the previous axiom. We are implicitly saying that the idealised bodies can be characterised by a sharp position, a point in Cartesian space and in addition that the rules of motion can be given for this point with independence of the rest of the properties of a more realistic description of everyday objects. We can anticipate that such point is the Centre of Mass, and will discuss later the restrictions that this idea introduces in the possible dynamics.

This (Cartesian) description includes some degree of objectivity in terms of coordinate transformations, since for any vector z^a within reference a, the following axiom holds.

AXIOM 2. *Let a and b be two different Cartesian reference frames describing the same relation in Nature, then $z^a = \sum_i \hat{\mathbf{e}}^a_i z^a_i = \sum_i \hat{\mathbf{e}}^b_i z^b_i$.*

The latter axiom is a consequence of imposing reasonability in the form of the No Arbitrariness Principle. As a consequence of it, independent directions and components transform as $\sum_i M_{ji}\hat{\mathbf{e}}^a_i = \hat{\mathbf{e}}^b_j$ and $\sum_i \left[M^{-1}\right]_{ij} z^a_i = z^b_j$, being M a non-singular matrix. If the choice is restricted to orthogonal directions the transformation $M \in GL(3)$ belongs in the orthogonal subgroup $O(3) \subset GL(3)$.

REMARK 1.1. The choice of (orthogonal) directions may be arbitrary but there is a group of transformations $\{R \in O(3)\}$, connecting any two choices, so that locations and distances are independent of the choice of R. The relative position between two objects becomes the vector $x_{AB} = x^a_B - x^a_A$ and their relative distance is $d_{AB} = |x_{AB}|$.

LEMMA 1.1. *The internal geometric relations of a static landscape (system) —i.e., where all bodies in the system are at rest relative to each other— are invariant with respect to the group formed by the semidirect product of the orthogonal* *group in three dimensions, $O(3)$, and the translation (Euclidean) group in three dimensions, $E(3)$, the symmetry group described as: $O(3) \ltimes E(3)$.*

Proof. The internal geometrical relations are given by relative distances and angles between vectors describing relative positions. The orientation vectors $\hat{\boldsymbol{e}}_i$ may be given by internal relations such as the direction, $\hat{\boldsymbol{e}}_1$, from cedar tree to birch tree. However, if the observer is moving, their relation to any point in the system will be changing by the same amount as the position of the original reference a with respect to the new point b. Even if observer a is moving with respect to the landscape, being the location of its reference point time-dependent as seen from a non-moving reference b (i.e, with x_a^b time-dependent), the relation of a to any point X in the system (namely x_X^a) will be changing by the same amount. Then $x_A^a = x_A^b - x_a^b = x_A^b + x_b^a$ and $x_B^b - x_A^b = x_B^a - x_A^a \equiv x_{AB}$, meaning that relative positions are invariant with respect to the choice of reference points. With respect to rotations, remark 1.1 shows that $\sum_i z_i^2 = \sum_j \left(z_j'\right)^2$ hence the distances (and in the same form the angles) between relative positions in the system are equal, being them referred to either system a or system b. □

Describing a static landscape with time-dependent reference points and axis orientations seems a bit awkward, since in the static case no difference whatsoever can be detected. However, understanding of motion is required for dynamics and will be presently addressed. In the same form, even in the case of a static scene moving relative to an observer, the internal geometry of the scene does not change when the description of the points in the system correspond to the change $x_Z^b = R(t)\left(x_Z^a - x_b^a(t)\right)$ where $x_b^a(t)$ is the position at time t of the reference point b as considered by a and $R(t)$ is a time-dependent rotation.

1.1.3. Relative velocity and time. Time is the word we use to express our perception of change; it is change in its most abstract form. Aristotle indicates "without change there is no time" [Book IV, Ch. 11, Aristotle, 1994–2010]. Piaget's experiments suggest that the notion of speed in the child precedes the notion of time:

> From the point of view of immediate experience, the child succeeds very soon in estimating speeds of which he has direct awareness, the spaces traversed in an identical time or the "before" and "after" in arrival at a goal in cases of trajectories of the same length. But there is a considerable gap between this and a dissociation of the notion of speed to extract a measurement of time, for this would involve replacing the direct intuitions peculiar to the elementary accommodation of thought to things by a system of relations involving a constructive assimilation [p. 383, Piaget, 1999].

No interval of time will ever be empty of changes. By comparing different regular phenomena we may convince us that the flow of time is uniform. From the point of view of our daily experience, as long as the Moon revolves around the Earth, we have change and we have time.

One striking difference between time and space from the psychogenetic point of view is that while for the change of position in space of an object an operation that reverts the change is conceivable, there is no operation capable to revert the temporal order of events. Unlike distance, that reaches us as a perception resulting from the telemetry of our binocular sight, time does not reach us as a perception, but it is rather the result of memory, a log of changes and a logical process that discriminates between the relative order of events. It follows that time is measured by comparing sequences of changes. By NAP, any transformation relating time perceptions of different references (egos) is constrained to preserve this ordering. Therefore, while arbitrary individual subjective time may differ among observers, they are all related by strictly monotonic (bijective) mappings. The underlying group is the set of strictly increasing continuous functions $f : \mathbb{R} \to \mathbb{R}$ with standard composition of functions as the group product.

AXIOM 3. *The order between the events (the various determinants and effects) involved in a causative relation is fixed.*

This axiom relates to the concept of "determinism" discussed around Equation (A) in [p. 199, Russell, 1912-1913]. In simpler words, first the vase falls, then it breaks. It is a vase as long as there exists a particular cohesion in the material. The alteration produced by the impact reorders the material in smaller pieces. Hence, without fall there is no impact, without impact no change in material stress. Causes here are gravitation and the change in material stress. Determinants are fall and impact, while breakup (rupture) is an effect.

Since we have an intuition of regular phenomena, we may agree on the functioning of reference clocks external to the process under study and hence consider time-records as real numbers with the usual sum operation. As long as we cannot present evidence that a process runs faster in one circumstance or another, we expect the relative order between changes in the clock and changes in the phenomena to remain the same, hence

ASSUMPTION 1. *An objective time can be defined by convening on a process to define a time-unit.*

Absolute time, a time encompassing all changes, appears to us in the same manner in which ego, not-ego, object and space emerge in the development of the child to produce a useful organisation of the world. Absolute time emerges in the construction of physics at the same step as a consequence of the same class of dialectical opening that creates the terms of an opposition that produces understanding. In this case the dialectic between system and environment (not-system) implies absolute time.

We finally consider the ratio between the rate of change for our (reference) clock and the change of relative positions,

$$v_{ij}(t_S, \delta) \equiv \frac{x_{ij}(t_S + \delta) - x_{ij}(t_S)}{(t_S + \delta) - t_S}$$

which reads as: the average velocity of i relative to j is the ratio of the difference between relative distances and the corresponding difference in clock records. If the ratio is independent of δ, then v_{ij} is constant. Thus, changes in position may also serve as a clock, sensing time intervals as spatial differences. Hence, time is as continuous as space is. It follows that

$$v_{ij}(t_S,\delta)+v_{jk}(t_S,\delta)=v_{ik}(t_S,\delta)$$

This is, whatever the clock is, the composition law between velocities must be the same that the composition law of the space.

1.2. Laws of motion

1.2.1. Abstraction, permanence and change in the laws of motion. In what follows we will use differential calculus freely. It is worth to recall that Newton and Leibniz developed it as the mathematical tool needed to describe mechanical motion. Mathematical calculus had to wait about 200 years in a somewhat precarious form until Karl Weierstrass resolved the mystery of the infinitesimal ϵ that notwithstanding being $\epsilon \neq 0$, it could be associated to $\epsilon^2 = 0$.

A law of motion is an abstraction in which all motions that differ only on conditions stated at a given time (usually the initial time, sometimes the final time and unusually some time in the middle of the time interval of interest) are considered to pertain to the same (abstract) class of motion. Correspondingly, any comparison with an idealised system must be performed specifying the law of motion and the "initial conditions".

The law must be constant in time, i.e., it must not change since we do not want what it is correct today to be wrong tomorrow, a situation that would invalidate experience.

Let x^a_A be the position of a body with respect to the reference system a, and let $x_{AB} = x^a_A - x^a_B$ be the relative position of different pairs of bodies as determined by a. Finally we write $x_{AB} = \sum_i \left(\left(x^a_{AB}\right)_i \hat{\mathbf{e}}^a_i\right)$ and simplify the notation by ignoring the reference a and changing capital letters to Greek letters for the bodies: $x_{\alpha\beta}$ stands for x^a_{AB} in the sequel. We cannot propose $\dot{x}_{\alpha\beta} = 0$ nor $\ddot{x}_{\alpha\beta} = 0$ as the law of motion because such laws can be easily refuted by observation. Our first serious candidate is $\dot{x}_{\alpha\beta} = v_{\alpha\beta}$, $\dot{v}_{\alpha\beta} = A_{\alpha\beta}(\{x_{\gamma\delta}\}, \{v_{\gamma\delta}\}, t)$ where the dot represents the derivative with respect to time and the indices $\alpha, \beta, \gamma, \delta$ run over a set of bodies (notice that we are interested only in some characteristic position, i.e., they are idealised as pointlike bodies).

However, we need the acceleration not to depend explicitly on time since otherwise we would have a different law at different times and it will fail to be universal. The law $\dot{x}_{\alpha\beta} = v_{\alpha\beta}$, $\dot{v}_{\alpha\beta} = A_{\alpha\beta}(\{x_{\gamma\delta}\}, \{v_{\gamma\delta}\})$ would be a suitable conjecture and it would become a proper belief if it survives the experimental tests.

The latter expression hides that $x_{AB} = \sum_i \left(x^a_{A,B}\right)_i \hat{\mathbf{e}}^a_i$, where the directions are not mathematical objects but symbols that indicate directions in the observable reality. If we want to write equations for the mathematical component of the displacement instruction, $\left(x_{\alpha\beta}\right)_i$ (which is notation for $\left(x^a_{AB}\right)_i$), we have to allow the observer a to pick their own references at will, something that includes the possibility of changing mind (continuously) with time. According to Remark 1.1 and Lemma 1.1 this action introduces an arbitrariness in the form of a time-dependent rotation, since $\frac{d}{dt}(R^a(t)z) = (R^a(t)\Omega^a \times z) + R^a(t)\dot{z}$ being Ω^a a time dependent angular velocity. If we let the acceleration be $A_{\alpha\beta} = \sum_i \left(\left(A^a_{\alpha\beta}\right)_i \hat{\mathbf{e}}^a_i\right)$, in terms of the positions observed by a with their choice of references, we have the law of motion:

$$
\begin{aligned}
\dot{x}_{\alpha\beta} - \Omega^a \times x_{\alpha\beta} &= v_{\alpha\beta} \\
\dot{v}_{\alpha\beta} - \Omega^a \times v_{\alpha\beta} &= A_{\alpha\beta}\left(\left\{x_{\gamma\delta}\right\}, \left\{v_{\gamma\delta}\right\}\right) \\
R^a(t) A_{\alpha\beta}\left(\left\{x_{\gamma\delta}\right\}, \left\{v_{\gamma\delta}\right\}\right) &= A_{\alpha\beta}\left(\left\{R^a(t)x_{\gamma\delta}\right\}, \left\{R^a(t)v_{\gamma\delta}\right\}\right)
\end{aligned}
\tag{1.2.1}
$$

where Ω^a is an arbitrary rotation vector and the last line indicates a condition that the components of the acceleration must satisfy. eq. (1.2.1) can be called the "law of subjective acceleration". notice that accelerations are equivariant under rotations but invariant under translations.

1.2.2. Newton's True motion. A central concept in Newton's Principia is the concept of *true motion*. Newton's discussion is obscured by mentions to absolute space that have attracted most of the attention of researchers, with a few exceptions (for example [DiSalle, 2020]). Newton writes [pp. 79- 80, Newton, 1687]

> ... absolute rest cannot be determined from the position of bodies in our regions... The causes by which true and relative motions are distinguished, one from the other, are the forces impressed upon bodies to generate motion. True motion is neither generated nor altered, but by some force impressed upon the body moved but relative motion may be generated or altered without any force impressed upon the body.

Newton's work is about True motion, in proof of that we offer the last paragraph of Chapter 1 just before he enters in the enunciation of his axioms:

> But how we are to collect the True motions from their causes, effects, and apparent differences, either true or apparent, and, vice versa, how from motions, either true or apparent, we may come to the knowledge of their causes and effects, shall be explained more at large in the following track. For to this end it was that I composed it. [p. 82, Newton, 1687]

True motion is actually the central concept that allowed J. J. Thomson to propose the concept of "Inertial frame of reference" [Thomson, 1884]. Textbooks and the exposition of knowledgeable people often forget the links with experience and the works of intuition. Let us give an example not in Newton's work. Consider the Olympic discipline known as "hammer throw". It consists in throwing a cannon ball as far as possible. Men use a 7.26kg ball while women use a 4kg ball. The ball is connected to a wire at most 122cm long which has a handle in the other extreme. The technique of throwing consists in having the ball revolving around the athlete so that it gains as much speed as possible and then letting it go for the mark. We are proposing here a real experience that any one doubting our description can experience by himself (unlike a "thought experiment" which cannot be compared with actual experience). Get the ball revolving around yourself as if you were an athlete and you will notice that the effort required is different depending on the mass of the ball and the speed you provide to it. You have to pull from the ball and incline your body so that you stay in balance. If you follow the ball with your eyes it is always in front of you. If you dare to look at some distant object that is not moving with respect to the ground, your visual perspective would suggest it is moving with respect to you[20]. In the reference system given by what is in front or behind you, left or right, above or below, the ball does not move and the grounded object moves (it may be a heavy construction for example). For a different observer, for example your trainer, the ball turns around you while at the same time you rotate around an axis along the vertical direction that intersects the ground about where your feet do. According to the trainer, the ball rotates because you are making an effort to change the direction of its velocity by pulling from the wire while the object fixed to the ground does not move. Thus, the visual description and the physical effort are in correspondence. A description that strictly matches the perceptions of the athlete could be that the ball is suspended in the air and objects fixed to the ground are moving, despite not knowing how they were influenced by the athlete. In this example, the trainer describes what Newton calls True motion and the athlete perceives "relative motion ... generated ... without any force impressed upon the body".

In terms of equations, considering eq. (1.2.1), the description of True motion corresponds to $\Omega = 0$. In this form we arrive to the form of Newton's second law, which is a law of True motion,

$$\begin{aligned} \dot{x}_{\alpha\beta} &= v_{\alpha\beta} \\ \dot{v}_{\alpha\beta} &= A_{\alpha\beta}\left(\left\{x_{\gamma\delta}\right\},\left\{v_{\gamma\delta}\right\}\right) \\ R^0 A_{\alpha\beta}\left(\left\{x_{\gamma\delta}\right\},\left\{v_{\gamma\delta}\right\}\right) &= A_{\alpha\beta}\left(\left\{R^0 x_{\gamma\delta}\right\},\left\{R^0 v_{\gamma\delta}\right\}\right) \end{aligned} \qquad (1.2.2)$$

(with R^0 a fixed rotation). It is always possible to write the equations of True motion if we have the equations of "relative motion" (which in Newton means apparent motion in the general case). Notice that we have the equation of motion with forms that do not depend any longer on a but the identification of a physical system of reference is pending, we address this task in Subsection 1.2.3. So far we have addressed the problem of the subgroup of arbitrariness produced by the arbitrary choice of the directions of reference and a residual arbitrariness has been left, represented by R^0 which corresponds to a time independent rotation.

Newton's meandering around the idea of "absolute space" has been an important source of problems. Absolute space relates directly with our original intuition as children and as such is appealing as a didactic transposition [Chevallard, 1989]. Newton rejected the idea of absolute space being a description based upon the distant stars. We read [p. 79, Newton, 1687]

[20] Be cautious because this experiment may promote dizziness, an error signal that the body sends when there are mismatches between sensors of equilibrium like in the condition known as motion sickness (dyskinesia).

> [...] For it may be that there is no body really at rest, to which the places and motions of other may be referred. But we may distinguish rest and motion, absolute and relative, one from the other, by their properties, causes and effects. It is a property of rest, that bodies at rest do rest in respect to one another. And therefore, as it is possible that in the remote regions of the fixed stars, or perhaps far beyond them, there may be some body absolutely at rest; but impossible to know, from the position of bodies to one another in our regions whether any of these do keep the same position to that remote body; it follows that absolute rest cannot be determined from the position of bodies in our regions.

Let us make a pause in the reconstruction to perfect the link with Newton's wording. In Newton, *relative motion* is the reordering of the spatial relations that we all recognise in response to (mostly) visual cues. *True motion* is a particular kind of relative motion in which we recognise the efforts that our motor skills demand. From effort Newton makes force and writes:

> The causes by which true and relative motions are distinguished, one from the other, are the forces impressed upon bodies to generate motion.[p.80, Newton, 1687]

However, True motion is not absolute motion as too many readers want to believe (we discuss this topic in the Scholium 1.A). Absolute motion is relative motion plus a conjectured universal reference for all motion (some times named the Alpha-body) that is at rest in absolute space[21]. As Newton argues (on page 79), the material existence of such body cannot be asserted. Newton, who recognises himself as Experimental Philosopher, obeyed the hygienic rule of reason: *hypothesis non fingo*, which in La Motte's translation reads:

> But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and **I frame no hypotheses**; for whatever is not deduced from the phenomena are to be called an hypothesis and hypotheses, whether metaphysical or physical, whether of occult physical, qualities or mechanical, have no place in experimental philosophy. [pp. 506 - 507, Newton, 1687] (Emphasis added).

It follows that for Newton to describe absolute motion was not only useless but incorrect as well. Further, notice that Newton addresses gravitation from phenomena, but phenomena are only accessible through intuition.

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[21] Neumann's postulate of an Alpha-body at rest somewhere in the universe has met criticism. However, DiSalle discussed the matter in a historical perspective [DiSalle, 1993]. Neumann's own account, as reported by DiSalle, deserves credit, he introduced the Alpha-body in the same form that a variable is sometimes introduced in mathematical arguments to exhibit more clearly the relations between other variables. The Alpha-body, an arbitrary body at rest, was introduced to share light into the relations among bodies known by experience. Neumann emphasised as well that its validity was equivalent to the "electrical fluid" and the "ether" that in his times (1878) were strong beliefs held by the physics community.

The great thinkers that considered that Newton argued in favour of absolute motion have to consider that Newton was an hypocrite. An alternative hypothesis is that they, as readers (or worse: indirect readers) might have not fully understood Newton. The second case just illustrates that when we read we receive visual cues that our brain organises in its own forms and not in the forms of the writer. Thus every time we produce and idea from the graph "rotating bucket" we make of it an idea which could be that of relative rotation in front of us (observer) and the surroundings, rotation relative to the fixed stars or true rotation (other alternatives are possible, such an allegedly existent centre of mas of the universe). The rotation with regard to the surroundings is objective (it can be perceived by our senses at least to the level of: "something has changed"). As Newton explains, True motion is an inference that explains observations, it is in principle an abduction (in Peirce's words), while motion with respect to the fixed stars is a fantasy, imagination, something that we can construct in our mind but it cannot be subject to experimental checks. Charles Peirce teaches that such hypothesis are not permitted in science, they are metaphysical: *hypothesis non fingo*.

1.2.3. Free motion and the principle of inertia. Let us now consider the True motion of free bodies. A *free body* is a body not influenced by other bodies or centres of action. Thus, considering true relative motion within a set of free bodies, their relative acceleration cannot depend on the relative distances to other bodies, nor on the relative velocities. If our set of bodies is a set of free bodies we have that $A_{\alpha\beta}\left(\left\{x_{\gamma\delta}\right\},\left\{v_{\gamma\delta}\right\}\right)=A_{\alpha\beta}$, being the rhs just a constant. However, a constant would indicate a particular direction in which two given free bodies should accelerate relative to each other in the coordinate system of reference a, which is arbitrary. Application of NAP using the orthogonal group allows us to arrive to the conclusion that such accelerations are zero, being zero the only vector invariant under rotations. The argument is equivalent to say that the universe is isotropic. In this presentation, Newton's first law of motion reads

THEOREM 1.1. *The true relative motion of free bodies corresponds to constant relative velocities.*

We realise then that if we take as reference the position of a free body and the reference axis along the direction of motion of other free bodies we have constructed an *inertial system of reference*, i.e., a reference system that describes True motion. However, the class of inertial systems would be of little help, unless we were able to find free bodies. The concept of free body can be viewed as an idealisation:

DEFINITION 1.2. **(Approximately free bodies)** A body is said to be *approximately free* for a given purpose if its interactions with other bodies can be ignored within the established level of tolerance. A *free body* is the idealised version of an approximately free body.

There is then an intuition of free bodies. Consider a game in which we have to throw marbles towards a target on a levelled (plane) playground. We have choices on the initial velocity that we can give to the marble to hit the target. If we consider the angle made by the initial velocity and the line that approximately connects our hand with the target at the time of releasing the marble, we can see that the larger the velocity the smaller the angle and the fastest the marble hits the target. We can then consider following Galileo'sidealisation that in the limit of infinite velocity the chosen angle will be zero as well as the time needed to reach the target. In such case, the gravitational attraction that makes the marble fall would have no time to act, the velocity would be constant along the trajectory, the motion straight. Since no force is effectively acting, such motion corresponds to free motion. Actual realisation of the experiment will produce approximately straight lines and we can get better approximations for greater velocities. Thus, what can be used as an inertial reference system depends upon our tolerance. This is actually what happened with the hammer thrower since we were in good disposition to accept that the translation of the Earth around the Sun and the rotation of Earth had no relevant influence for our discussion, a reference frame that is fixed to the surface of Earth could be considered an inertial system for the purpose.

1.2.4. True motion in Cartesian space. We can now rewrite the prototype of the equation of motion in a form closer to the one used by Newton and found in all textbooks we have examined. Let us use as reference a set of free bodies, and let us simplify the notation dropping the index of the free body A that acts as reference. In the space of this set of free bodies (notice that we need at least three bodies to determine the origin and the reference directions[22]), eq. (1.2.2) reads

$$\begin{aligned} \dot{x}_\beta &= v_\beta \\ \dot{v}_\beta &= A_\beta\left(\{x_{\gamma\delta}\},\{v_{\gamma\delta}\}\right) \\ R^0 A_\beta\left(\{x_{\gamma\delta}\},\{v_{\gamma\delta}\}\right) &= A_\beta\left(\{R^0 x_{\gamma\delta}\},\{R^0 v_{\gamma\delta}\}\right) \end{aligned} \tag{1.2.3}$$

"The position of β is x_β" is only a shorthand for "the position of β relative to α in the frame of reference a centered in A (i.e., α) and with axes of directions given elsewhere is x_β". The utility of the shorthand is obvious but there is a danger using it when the full expression is not clearly imprinted in the mind of the reader, since it connects with simple intuitions of space acquired in their early days of life.

In the same form, an objective relative velocity can be established as the change in relative position between A and B from time t_0 to time t in the form

$$v_{AB} = \lim_{t\to t_0} \frac{x_{AB}(t) - x_{AB}(t_0)}{t - t_0}$$

(for free bodies the expression is constant and it is its own limit). The velocity of a (not necessarily free) body B with respect to the origin, O (which coincides with a or α) is then

$$v_{OB} = \lim_{t\to t_0} \frac{x_{OB}(t) - x_{OB}(t_0)}{t - t_0}$$

[22] Except in exceptional situations (full alignment or zero relative velocities) the three bodies make for two independent directions, say: x_{AB}, x_{BC}, while $x_{CA} = -\left(x_{AB} + x_{BC}\right)$. There are also two independent relative velocity directions. It is enough for one of them not to belong to the plane containing ABC to have a third independent reference direction.

This change be can written dropping the reference to the origin to lighten the notation. In such a case it appears to be a property of B when it actually is a property of $\{O, B\}$. Since $x_{AB} = x^a_B - x^a_A = x^b_B - x^b_A$ as long as the reference frames a and b differ only in their locations, we can retain the expression in a and drop the index, $x_{AB} = x_B - x_A$. Finally, we introduce all the symbol neglecting operations and write

$$v_B = \lim_{t \to t_0} \frac{x_B(t) - x_B(t_0)}{t - t_0},$$

which reads: the velocity of B is the limit value of B's change of position in space operated between times t_0 and t divided by the time interval. The operation is exactly equivalent to realise that $x_{OO}(s) = 0\, \forall s$ (actually, it holds for any body/reference). The child in us smiles: I knew it!, but the velocity v_B is subjective. It is the calculation of a subject so habituated to be the centre of the Universe that has made it "natural" (habitus).

Definition 1.3. *Objective velocity* is the rate of change of relative position between two bodies with respect to the change in time,

$$v_{AB} = \frac{d(x^a_A - x^a_B)}{dt} = \lim_{\Delta t \to 0} \frac{\big(x^a_A(t + \Delta t) - x^a_B(t + \Delta t)\big) - \big(x^a_A(t) - x^a_B(t)\big)}{\Delta t}.$$

1.2.5. Subjective velocity. The relative velocity between the reference point chosen by the observer a and a body A, follows Definition 1.3,

$$v_{Aa} = \frac{d(x^a_A - x^a_a)}{dt} = \lim_{\Delta t \to 0} \frac{\big(x^a_A(t + \Delta t) - x^a_a(t + \Delta t)\big) - \big(x^a_A(t) - x^a_a(t)\big)}{\Delta t}.$$

We now focus on the subjective operation consisting in setting $x^a_a(t+\Delta t) - x^a_a(t) = 0^a$ and similarly $v^a_a = \lim_{\Delta t \to 0} \frac{x^a_a(t+\Delta t) - x^a_a(t)}{\Delta t} = 0^a$, this is to say that for the observer, the point designated as reference by her/his arbitrary decision does not move (we have added the superscript a to the zero to indicate the subjectivity). Now the relative velocity reads

$$v_{Aa} = \frac{d(x^a_A - x^a_a)}{dt} = \lim_{\Delta t \to 0} \frac{x^a_A(t + \Delta t) - x^a_A(t)}{\Delta t} - 0^a \equiv v^a_A - 0^a,$$

where we call v^a_A the *subjective velocity* of A as established by observer a and correspondingly $x^a_A(t + \Delta t) - x^a_A(t)$ is the *subjective distance* travelled by A.

To speak of bodies occupying positions in space and having velocities and accelerations contributes to the acceptance of formulae like eq. (1.2.3). However, hiding the assumptions adopted to attain such kind of expressions will prove to be dangerous as ignorance always is.

1.2.6. Galileo's transformations. Despite our efforts selecting a reference system that reveals true motion, there is still ambiguity in it. It is almost trivial to show that any transformation of the form:

$$y = R^0\,(x + d)$$

where R^0 is a fixed rotation and $d = ut + d_0$ with u and $d_0$time independent vectors, produces equations of the same form than eq. (1.2.3). The translations subgroup $E(3)$, of this group of transformations $(O(3) \ltimes E(3))$ has been called Galileo's (or

Galilean) group. Conceptually, it is the subset of the geometric group restricted to reference systems that display True motion.

COROLLARY 1.1. *For $R^0 = Id$ and $d_0 = 0$, if a is an approximately free reference and $x_A^b = x_A^a + v_{ab}t$ (with v_{ab} constant) then b is also an approximately free reference.*

1.2.7. Gravitation and mass. To understand how gravitation is handled in Newton's mechanics we must begin with some experiences that the reader can almost certainly perform. Newton derived the concept of mass from density and volume, in his first definition

> The quantity of matter is the measure of the same, arising from its density and bulk conjunctly. [p. 73, Newton, 1687]

By "conjunctly" he means as a product

$$m = \rho V$$

that reads "mass is the product of density and volume". He further explains that it is proportional to the weight, a conclusion he reached using very accurate pendulums.

Archimedes principle has been know since circa 200 BC, thus densities can be measured with some accuracy by disregarding the density of the air with respect to the density of a liquid (say water), which is an error of about 1/1200. We can easily see that bodies of similar volume but made with different materials fall in water at different velocities (some of them like most woods move up rather than falling). If we restrict our attention to those bodies that fall in water we can see that they reach the ground "at the same time" (with our capabilities of observation) when released from a high point (tradition wants it to be the Leaning Tower of Pisa). There appears to be no dependence of the acceleration with the mass of the falling body in the absence of substantial push exerted by the displaced fluid.

The Roman (Steelyard) Balance is also known since 200 BC and works on Archimedes "law of the lever". It admits a variation attributed to Galileo known as "hydrostatic balance" that allows for the direct measurement of densities. Thus, if

$$m_B \propto W_B,$$

$A_{EB} \propto \frac{W_B}{m_B} \equiv g$ can be idealised as independent of the body, and we can conceive Earth, E, as "just another body" (A_{EB} is a candidate for the acceleration imparted to body B by interaction with the Earth), we have the problem that the lhs is symmetric with respect to the exchange of bodies and the rhs is not. When we recognise that all the experience at hand comes from probing the law using different bodies but leaving always the Earth as the interacting partner, we can see we have as much evidence for the latter relation as we have for $m_B A_{EB} \propto W_B = \frac{g}{m_E} m_E m_B$ whose rhs is symmetric iff $\frac{g}{m_E} = G$ is constant under the exchange of other bodies with the Earth. The relation $G m_E m_B f(x_E - x_B)$ can account for the situation with other matter not tested. The vector $f(x_B - x_E)$ represents the unknown dependence on relative distance, yet it must transform in front of rotations as $x_B - x_E$, an observation that further narrows the set of candidates for the gravitational force. The no arbitrariness principle acts again to perfect and complement our construction of physics. The proposed relation would explain why massive bodies nearby the falling objects do not seem to affect their motion, something to be expected by the differences in volume and the the perception that the bodies being used are made out of matter taken from Earth.

A contemporary of Newton, Robert Hooke, had explored the relation between the elongation of springs and weight. When Newton presented his work to the Royal Society, Hooke complained that he had the priority on the discovery of the gravitation law [Life of Sir Isaac Newton, Newton, 1687]. The controversy [Guicciardini, 2005] was sustained in material presented to the Royal society and, hence, it is well documented. Newton acknowledges Hooke's work in [p. 108, Newton, 1687]. Among the allegations made by Newton he indicated that the idea of a force that weakens with distance had been entertained by other philosophers or astronomers, not just Hooke, and mentioned Ismaël Bullialdus (France, 1605 – 1694) and Giovanni Alfonso Borelli (1611 – 1679). While the idea of a dependence of force with distance was indeed entertained by several authors, only Newton put it into formulae and showed the consequences of accepting it as a universal law.

Given the form of the universal gravitation it seems convenient to consider $m_B A_{EB}$ rather than A_{EB} and write the equations of motion introducing the "quantity of motion" defined as (second definition):

> The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjunctly.

In modern notation

$$p = mv$$

Since the mass, m, is a property of the body, there is no reason to believe that it can change with time, hence the eqs. (1.2.3) transform easily into Newton's second law.

1.2.8. The concept of force. As far as (local) gravitation is concerned we have that

$$m_B A_{EB} = -m_E A_{BE} = W_B$$

The concept of weight carries more generality than that of acceleration. Weight relates with our senses directly and we have a simple intuition for it, hence, we know what it means.

The study of other means of producing motion and of equilibrating weight with some other counteraction is a form of studying entities that belong qualitatively to the family of weight. By the times of Newton, Hooke had set his rule on the "potentia restitutiva" of springs, or "the power of springing bodies", in the form:

> About two years since I printed this Theory in an Anagram at the end of my Book of the Descriptions of Helioscopes , [...]; That is, The Power of any Spring is in the same proportion with the Tension thereof: That is, if one power stretch or bend it one space, two will bend it two, and three will bend it three,

> and so forward. Now as the Theory is very short, so the way of trying it is very easie.
>
> Take then a quantity of even-drawn Wire, either Steel, Iron, or Brass, and coyl it on an even Cylinder into a Helix of what length or number of turns you please, then turn the ends of the Wire into Loops, by one of which suspend this coyl upon a nail, and by the other sustain the weight that you would have to extend it, and hanging on several Weights observe exactly to what length each of the weights do extend it beyond the length that its own weight doth stretch it to, and you shall find that if one ounce, or one pound, or one certain weight doth lengthen it one line, or one inch, or one certain length, then two ounces, two pounds, or two weights will extend it two lines, two inches, or two lengths; and three ounces, pounds, or weights, three lines , inches , or lengths; and so forwards [p. I, Hooke, 1678]. [Some "long s" characters have been replaced by "s" since in the original there is little typographical difference between "f" and "long s"]

In modern notation, Hooke's law reads

$$f = k\Delta l$$

where f is called the force, k characterises the spring and Δl is the elongation from the position of rest when nothing hangs from the spring. The "potentia restitutiva" or "power of the spring" became "the force" and it is gauged by comparison with weight. Thus

Definition 1.4. Force is the name of the mechanical causative action that belongs in the same class than weight (at least qualitatively). Forces can counteract weights and can produce accelerations as much as weight does.

Later, other forces will follow an equivalent path, for example Coulomb's measurements of electrostatic forces with his balance. Also Ampère measured forces between conductors carrying currents using static balances [Ampère, 1823].

In this form we see that the concept of force cannot be separated from the concept of weight. Force is defined statically, and a hidden principle operates. Newton writes:

> A body by two forces conjoined will describe the diagonal of a parallelogram, in the same time that it would describe the sides, by those forces apart.[p. 84, Newton, 1687] (Corollary I in chapter "Axiom or laws of motion")

What Newton does not say is that to prove the Corollary the assumption that the conjoint action of (independent) forces corresponds to the addition of the forces is needed.

Definition 1.5. Two forces are said to be independent when the resulting acceleration produced by both of them conjointly is the sum of the accelerations produced by each force acting in the absence of the other.

Newton's corollary results from the second law of motion for independent forces.

The forces are called independent because the action of any one does not interfere with the causative action of the others. It follows that if two independent forces can be balanced resulting in a null acceleration (equilibrium) they are of equal modulus but have opposite directions. We then have a method to compare and gauge forces. Actually, we can measure densities, and masses, with Galileo's balance because we assume without saying that the pushing of the displaced fluid does not interfere with the pulling of gravitation, which is something intuited.

We soon realise that when we speak about masses we need to introduce conjointly acting causes of motion, named forces, and if we are to assign values to such masses we have to resort to means of comparison of forces and to a formula representing their relations. Then, mass, force and Newton's second law emerge together and the terms make sense only in relation to the other terms. The second law is in part synthetic, because the concept of acceleration exists without any need for dynamics but it is a dialectical opening in as much mass and force cannot be conceived except as a relation of one with the other. They do not preexist Newton's law. Hooke's paper from 1678 is an example that the word force had not acquired in those times the meaning it has now.

1.2.9. Action and reaction of instantaneous action at distance. Newton's second law reads so far

$$\begin{aligned} \dot{x}_\beta &= v_\beta \\ m_\beta \dot{v}_\beta = \dot{p}_\beta &= F_\beta\left(\{x_{\gamma\delta}\},\{v_{\gamma\delta}\}\right) \\ R^0 F_\beta\left(\{x_{\gamma\delta}\},\{v_{\gamma\delta}\}\right) &= F_\beta\left(\{R^0 x_{\gamma\delta}\},\{R^0 v_{\gamma\delta}\}\right) \end{aligned} \tag{1.2.4}$$

where F stands for force and we keep the reminder on how force must transform under a fixed rotation. We had that by virtue of the law being written as relations to a reference system composed of free bodies, what the observed bodies are making one to the other cannot affect the motion of a free body, for otherwise it would be *not free*.

Playing "tug war" (rope pulling) can easily convince us that when the opponent stops pulling we are put into motion, hence the absence of motion is related to an equilibrium of forces, and these are the same kind of forces we use to move up a body against gravitational forces (weight). The same experience corresponds to all contact forces: if we are in equilibrium, what we are pushing also pushes us with a force that is just minus the force we are exerting. Thus, simple intuition tell us that the motion of the contact point does not change whenever forces are balanced. The same can be said of the gravitational force, since $f(x_E - x_B) = (x_E - x_B)\ h(|x_E - x_B|)$: the force exerted by E over B is equal in value but has opposite direction than that exerted by B over E, hence, the sum of both forces cancels.

A substantial portion of Newton's *Principia* is dedicated to characterise gravitation. He argues that the same *universal gravitation* lies behind free fall near the surface of the earth, the tides of the seas, the motion of the Moon around the Earth as well as that of the planets and comets around the Sun [p.107-108, Prop. IV Theo. IV; p. 385, last paragraph of Rule III; Newton, 1687]. By studying planetary orbits as well as free fall near the Earth [Escholium, p. 89, Newton, 1687], he concludes that the interaction depends only on the relative position of the interacting bodies (namely that in $f(x_A - x_B)$ above we have $h(|x_A - x_B|) = \gamma |x_A - x_B|^{-3}$, with γ a universal constant), being therefore of instantaneous action. Moreover,

COROLLARY 1.2. *The total momentum* $p_A + p_B \equiv m_A v_A + m_B v_B$ *is constant in time when two (otherwise free) bodies interact gravitationally.*

PROOF. By the second law, we have that

$$\begin{aligned} \frac{d}{dt}(p_A + p_B) &= \frac{d}{dt}(m_A v_A + m_B v_B) \\ &= G m_A m_B \left(f(x_A - x_B) + f(x_B - x_A) \right) = 0 \end{aligned}$$

□

There is then a temptation to introduce the third law:

> To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.[p. 83, Newton, 1687]

"Contrary parts" refers to contrary parts of space and it is not necessary for forces to be "central", i.e., to be exerted in the direction joining the bodies (that would have to be assumed pointlike in such a case).

The quantity $(m_A v_A + m_B v_B) = \frac{d}{dt}(m_A x_A + m_B x_B) = (m_A + m_B) v_{CM}$ is written as the momentum of the centre of mass, $x_{CM} = \dfrac{m_A x_A + m_B x_B}{m_A + m_B}$. Thus, the assumption made when considering the location of bodies as represented by a point in the space is consistent with the Laws of motion. The complementary idea that the internal state of motion is decoupled from the motion of the representative point x_{CM} is instrumented in the third law. Thus, the forces that hold together material objects have to satisfy Newton's third law for otherwise they would experience something like self-induced motion. In Newton's time these forces were gravitation and the mysterious force that holds together the common material objects.

Unlike the first and the second law that are the result of a philosophical intuition (intuition supervised by reason), the third law is closer to raw empiricism, it is as good as "all swans are white" but plays a substantial role in the aggregation of parts to conform a larger body. In terms of possible universal laws for reciprocal action (interaction) it places severe limitations to them. Gravitation, as described by Newton satisfies the third law.

The debate about the speed by which gravitation propagates has lasted for years. In the context of Newton's dynamics and gravitational forces, experiments set a lower limit for such speed of 2×10^{10} times the speed of light [Van Flandern, 1998, Marsh and Nissim-Sabat, 1999, Van Flandern, 1999]. However, the limit says nothing if the context is not that of forces and Newton's dynamics. To abandon instantaneous action at distance means to abandon Newton's schema which in turn requires to construct a new (and better) schema that links to our intuitions. Failing to constructively link the schema to the intuitions would throw us into an epistemic change where the meaning of “correct” has to be changed. When a new epistemic praxis is consecrated as correct, and gravitation is assumed to propagate with finite velocity, not only what we have learned comparing forces would have to be dropped as it derives from the “wrong” forms of thinking but the complete form of Galileo’s approach would have to be dropped as we drop phenomenology, which is the “art” of putting in relation what we observe with what we think under the supervision of reason, in the case of physics transcribed in mathematical language.

We are not saying that it is an impossible task, what we indicate is that to sustain the old scholastic lemma held against Newton: “matter cannot act where it is not”, or using Newton’s equations outside their validity range in order to prove Newton wrong are invalid forms of reasoning that might bring all the scholastic attitude with it. Indeed, we will explore in the coming chapters how physics returned to the medieval attitude of sustaining an indisputed doctrine, with a clergy of versed scholars guarding the sacred beliefs.

1.2.10. Limitations of the concept of force and other interactions. So far, Newton’s three laws hold for gravitation and forces that can balance gravitation (such as those idealised from elongated springs or tense ropes). Regardless of the actual form of the interactions occurring in Nature, some of their properties can be derived by application of NAP, before considering the specific forms. We will consider in this section interaction between two pointlike particles through forces of instantaneous action (not depending explicitly on time), satisfying the additive property of Definition 1.5 and eq. (1.2.4). This last requirement means that such a force f is the *cause* of acceleration i.e., the totality of the interaction is exhausted in the acceleration (which is the ratio between force and mass). These assumptions have to be contrasted with experiments for any “new” force. We have

$$\begin{aligned} m_A \frac{d^2}{dt^2} x_A &= f(Q_A, x_A, v_A, Q_B, x_B, v_B, ...) \\ m_B \frac{d^2}{dt^2} x_B &= k(Q_A, x_A, v_A, Q_B, x_B, v_B, ...) \end{aligned}$$

where f is the force exerted by B on A, k is the corresponding force exerted by A on B, and Q is the quantity of the physical substance with additive properties quantifying the forces (think e.g., of electric charge and the Coulomb force). The dots stand for other possible external dependencies. In relative coordinates we have, letting $x_{CM} = \frac{m_A}{m_A + m_B} x_A + \frac{m_B}{m_A + m_B} x_B$ and $x_{AB} = x_A - x_B$ (the relative position)

$$\begin{aligned} (m_A + m_B) \frac{d^2}{dt^2} x_{CM} &= f(Q_A, Q_B, x_{AB}, v_{AB}, ...) + k(Q_A, Q_B, x_{AB}, v_{AB}, ...) \\ \left(\frac{m_A m_B}{m_A + m_B} \right) \frac{d^2}{dt^2} x_{AB} &= \frac{m_B f - m_A k}{m_A + m_B} \end{aligned} \tag{1.2.5}$$

The laws of Nature cannot depend on the choice of labels for particles A and B. Hence, LEMMA 1.2. *Let T be the operation that interchanges the defining properties Q_A, Q_B of a pair of interacting particles and f, k be the additive forces enacting the interaction. Then, the NAP symmetry constraints demand*

$$\begin{aligned} Tf(Q_A, Q_B, x_{AB}, v_{AB}, ...) &= f(Q_B, Q_A, x_{AB}, v_{AB}, ...) \\ &= k(Q_A, Q_B, -x_{AB}, -v_{AB}, ...) \quad (1.2.6) \\ Tk(Q_A, Q_B, -x_{AB}, -v_{AB}, ...) &= k(Q_B, Q_A, -x_{AB}, -v_{AB}, ...) \\ &= f(Q_A, Q_B, x_{AB}, v_{AB}, ...). \quad (1.2.7) \end{aligned}$$

and $T^2 = Id$.

The most general expression for a vector force depending on vectors x_{AB} and v_{AB} reads,

$$f = x_{AB}\phi_d + v_{AB}\phi_s + (x_{AB} \times v_{AB})\,\phi_\perp$$

where ϕ_d, ϕ_s, $\phi_\perp$ are scalar functions of $(Q_A, Q_B, x_{AB}, v_{AB}, ...)$.

COROLLARY 1.3. *By Lemma 1.2 the interacting forces f, k satisfy the* generalised principle of action and reaction, *namely that given f as above, then*

$$k = -x_{AB}\phi_d - v_{AB}\phi_s + (x_{AB} \times v_{AB})\,\phi_\perp.$$

LEMMA 1.3. $\phi_\perp = 0$.

PROOF. Interchanging the attributes $m_A, Q_A \leftrightarrow m_B, Q_B$ from eq. (1.2.5) we obtain, by repeated application of eqs. (1.2.6) and (1.2.7) in Lemma (1.2),

$$\begin{aligned} \left(\frac{m_A m_B}{m_A + m_B}\right)\frac{d^2}{dt^2}x_{AB} &= \frac{m_A}{m_A + m_B} f(Q_B, Q_A, x_{AB}, v_{AB}, ...) \\ &\quad - \frac{m_B}{m_A + m_B} k(Q_B, Q_A, x_{AB}, v_{AB}, ...) \\ &= \frac{m_A}{m_A + m_B} k(Q_A, Q_B, -x_{AB}, -v_{AB}, ...). \\ &\quad - \frac{m_B}{m_A + m_B} f(Q_A, Q_B, -x_{AB}, -v_{AB}, ...). \end{aligned}$$

The lhs is invariant and hence the rhs must be so as well. Therefore, $f(Q_A, Q_B, -x_{AB}, -v_{AB}, ...) = -f(Q_A, Q_B, x_{AB}, v_{AB}, ...)$ and also $k(Q_A, Q_B, -x_{AB}, -v_{AB}, ...) = -k(Q_A, Q_B, x_{AB}, v_{AB}, ...)$. This in turn requires ϕ_d, ϕ_s to be invariant in front of rotations and reflexions while $\phi_\perp$ must change sign upon reflexion, still being a scalar function of the vectors x_{AB}, v_{AB}. However, $x_{AB} \cdot v_{AB}, x_{AB}^2, v_{AB}^2$ are all reflexion invariant scalar functions, hence $\phi_\perp = 0$. □

In the sequel, we drop all references to $\phi_\perp$.

REMARK 1.2. Gravitation, as conceived by Newton, as well as Coulomb's electrostatic force between electrically charged bodies correspond with $\phi_s = 0$. Interaction forces satisfying $\phi_s = 0$ are called *central forces.*

COROLLARY 1.4. *Under the Generalised Principle of Action and Reaction the total momentum, $m_A v_A + m_B v_B$ is constant in time.*

PROOF. Writing out f and k, we have

$$\begin{aligned}\frac{d}{dt}\left(m_A v_A + m_B v_B\right) &= f + k \\ &= x_{AB}\left(\phi_d - \phi_d\right) + v_{AB}\left(\phi_s - \phi_s\right) = 0\end{aligned}$$

□

COROLLARY 1.5. *Under the Generalised Principle of Action and Reaction, if* $\phi_s = 0$ *the interaction forces produce no internal torque.*

PROOF. The internal torque satisfies

$$\begin{aligned}x_{AB} \times \left(\frac{m_A m_B}{m_A + m_B}\right) \frac{d^2}{dt^2} x_{AB} &= x_{AB} \times \frac{m_B f - m_A k}{m_A + m_B} \\ &= \left(x_{AB} \times v_{AB}\right) \phi_s\end{aligned}$$

□

COROLLARY 1.6. *If* $\phi_s = 0$ *and* ϕ_d *depends only on* $|x_{AB}|$, *the internal energy* $E = \frac{1}{2}\left(\frac{m_A m_B}{m_A + m_B}\right) v_{AB}^2 + V(|x_{AB}|)$ *is constant in time, where* $x_{AB}\phi_d = -\nabla V(|x_{AB}|)$.

PROOF. We have that

$$\begin{aligned}v_{AB} \cdot \frac{d}{dt} v_{AB} \equiv \frac{d}{dt}\left(\frac{1}{2} v_{AB}^2\right) &= v_{AB} \cdot \left(\frac{f}{m_A} - \frac{k}{m_B}\right) \\ &= v_{AB} \cdot \left(-\nabla V(|x_{AB}|)\right)\left(\frac{1}{m_A} + \frac{1}{m_B}\right).\end{aligned}$$

Noticing that $-\frac{dV}{dt} = -\nabla V(|x_{AB}|) \cdot v_{AB}$ we arrive at

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{m_A m_B}{m_A + m_B}\right) v_{AB}^2 + V(|x_{AB}|)\right) = 0.$$

□

The discussion illustrates how a systematic use of NAP limits the possible formulation of forces. However, a second set of forces can be systematically treated, they correspond to those forces indicated by their effects, usually known as constraints.

1.2.11. A note on Lagrange's formulation. The treatment of the forces associated to constraints of the motion follows a historical development that begins with intuition: if a body is constrained to move on a surface, the force exerted by the surface on the body is perpendicular to the surface, i.e., as long as the constraint is operative, the body can neither leave the surface (just "flying around") nor break its way through it; the constraint compensates exactly other forces operating along the perpendicular direction. If the surface is described by an equation of the form $\phi(x;t) = 0$ with $x \in \mathbb{R}^{3N}$ and N the number of bodies in consideration, the force reads $F = \lambda\nabla\phi$ since $\nabla\phi$ lies along the normal vector $\hat{n}$ to the surface at each point. If an infinitesimal change in initial conditions were to produce an infinitesimal change, δ, we would have $\phi(x+\delta,t) \sim \nabla\phi \cdot \delta = 0$, since δ lies on the surface and $\nabla\phi$ along the normal vector. The displacement δ is named a *virtual displacement* and the relation indicates that forces deriving from this sort of geometrical constraints do not perform "virtual work", which is D'Alembert's principle.[23]

LEMMA 1.4. *If a force satisfies D'Alembert principle the force is normal to the constraining surface.*

PROOF. We write $F = f\hat{n} + G$ with $G \cdot \hat{n} = 0$, according to D'Alembert's principle $\delta \cdot F = 0 = G \cdot \delta$. The displacement δ spans a subspace of dimension $3n-1$ being its only constraint that $\delta \cdot \hat{n} = 0$. Hence, the projection of G on the subspace perpendicular to $\hat{n}$ must be zero, but $G \cdot \hat{n} = 0$, then $G = 0$. □

The next step was made by Lagrange and it can be thought of in different forms. Quite often textbooks obscure the issue (as for example [Goldstein, 1980]) by reading the mathematics in $\phi(x+\delta,t) \sim \nabla\phi \cdot \delta = 0$ by carrying out the "virtual displacement", an artifice in which the system is frozen at time t and the coordinates change by δ before the system is allowed to continue the evolution. Having suppressed Lemma 1.4 and the relation of the virtual displacements with the initial (intermediate or final) conditions, the elaboration becomes mathematically formal and Hegel's diatribes apply directly (see quotation 1.A.4). Other authors go further into empty forms and simply declare Hamilton's principle the starting point of mechanics [Landau and Lifshitŝ, 1982].

Our presentation can be found in Hamilton [1834] seminal paper. We will not complete Lagrange's deduction, yet we indicate that if we have a set of independent constraints $\phi_\alpha(x,t) = 0$ with $\alpha = 1 \ldots N_c < 3N$ then, the space locally decomposes in the N_c directions $\hat{n}_\alpha$ (independent directions, which is the meaning of "independent constraints") along which the constraints prohibit motion and a set of $3N - N_c$ local coordinates along which the system can move. If we manage to produce such "generalised" coordinates in the form $q_i = q_i(x,t)$ for $i = 1, \cdots, 3N - N_c$, Newton's equations can be split into $(3N - N_c)$ equations of motion and N_c equations for the forces associated to the constraints as a function of positions, velocities and accelerations).

Lagrange introduced a functional, today known as Lagrangian, which depends on the generalised coordinates q_i, their associate time-derivatives (the "generalised velocities") ν_i–, and time

$$\mathcal{L} = \mathcal{L}(\{q_i\}, \{\nu_i\}, t)$$

that encode the equations of motion in the form

$$\begin{aligned} \frac{d}{dt} q_i &= \nu_i \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} &= 0 \end{aligned}$$

[23] A course note on Lagrange's formulation (in Spanish) along these ideas is available as http://dx.doi.org/10.13140/RG.2.2.19310.32322/1.

The Lagrangian for a system of pointlike bodies interacting by explicit forces that admit a scalar potential, $f_i = -\nabla V(\{x_i\}, t)$ reads

$$\mathcal{L} = T - V$$

with $T = \frac{1}{2}\sum_{j=1}^{3N}\left(\frac{dx_j}{dt}\right)^2$ the kinetic energy, that must be rewritten in terms of the "generalised velocities and coordinates" q_i, ν_i.

Lagrange's formulation is a generalisation of Newton's mechanics, it produces Newton's equations when no constraints are given and allows for more general relations.

The Lagrangian is not unique in the sense that different Lagrangians can produce the same equations of motion. This property results in a problem when physics is read from equations, a practice known as "interpretation". Let us offer two non-uniqueness results (they are not the most general, but are sufficient for our purpose).

THEOREM 1.2. *Two Lagrangians that differ by*[24] $U = \Phi_{,t} + \sum_i \nu_i \Phi_{,q_i}$ *for any twice differentiable function* $\Phi(\{q_i\}, t)$ *produce the same equations of motion.*

PROOF. It suffices to show that the contribution of U to the dynamical (Euler-Lagrange) equations is identically zero. Consider the contribution of U to the equations of motion:

$$\begin{aligned}\frac{d}{dt}q_i &= \nu_i \\ \frac{d}{dt}\left(\frac{\partial U}{\partial \nu_i}\right) - \left(\frac{\partial U}{\partial q_i}\right) &= \frac{d\Phi_{,q_i}}{dt} - \frac{\partial\left(\Phi_{,t} + \sum_j \nu_j \Phi_{,q_j}\right)}{\partial q_i}\end{aligned}$$

and the second equation is identically zero given the first one. □

Finally, let us consider a gravitational problem in which a point particle of mass m moves in the presence of a distribution of mass given by the density $\rho(x)$ in a Cartesian space that privileges the description of the mass distribution (not an inertial system of reference). The gravitational potential associated with the distribution of masses, V, satisfies Poisson's equation:

$$\Delta V = -\frac{\rho}{4\pi}$$

with boundary conditions for $\lim_{|x|\to\infty} V(x) = 0$. We introduce a wave operator associated with waves of velocity a in the form

$$\Box_a = \Delta - \frac{1}{a^2}\partial^2_{tt}$$

and a "free" wave Φ that satisfies

$$\Box_a \Phi = 0$$

We let $A = \nabla\Phi$ and $U = \Phi_{,t}$

[24] $\Phi_{,y}$ denotes partial derivative of Φ with respect to y.

COROLLARY 1.7. *Lagrangians $\mathcal{L} = T - V$ and $\mathcal{L}' = \mathcal{L} + \nu \cdot A + U$ have the same dynamical equations*

PROOF. It suffices to notice that when $\frac{dx}{dt} = \nu$,

$$\nu \cdot A + U = \frac{d\Phi}{dt}.$$

Hence, both Lagrangians differ in the total derivative of a function Φ, and this does not alter the dynamical equations. $\square$

However, if we write $V' = V - U$ we have that

$$\Box_a V' = -\frac{\rho}{4\pi}$$

that naively can be misinterpreted as a propagation of the gravitational potential with arbitrary velocity a. The invariance of the equations of motion with respect to these transformations is known as *gauge symmetry* and it is not a physical property but a property of the Lagrangian formulation that will come along with every physical law formulated[25] using Lagrangians. The best known example of this approach is electromagnetism (see Chapter 4).

[25] Meaning written as mathematical relations.

Appendix

1.A. Scholium. Criticism of scholastic mechanics

1.A.1. Introduction: On the Enlightenment approach to understanding. The method we have followed in this chapter tries to make clear what we think before reading what others think. We propose a direct dialog with Nature, an experience in which we have been engaged since we were born. The method, although time consuming, leads towards independence of thought and relates to W. von Humboldt's concept of "bildung"[Sorkin, 1983]. The aim is to acquire our own, independent, idea of how Nature works, something that is required if we want to achieve "Enlightenment" which according to Kant means:

> 1. Enlightenment is man's emergence from his self-imposed immaturity. Immaturity is the inability to use one's understanding without guidance from another. This immaturity is self-imposed when its cause lies not in lack of understanding, but in lack of resolve and courage to use it without guidance from another. *Sapere Aude!* "Have courage to use your own understanding!" -- that is the motto of enlightenment.
>
> 2. Laziness and cowardice are the reasons why so great a proportion of men, long after nature has released them from alien guidance (*natura-liter maiorennes*), nonetheless gladly remain in lifelong immaturity, and why it is so easy for others to establish themselves as their guardians. It is so easy to be immature. If I have a book to serve as my understanding, a pastor to serve as my conscience, a physician to determine my diet for me, and so on, I need not exert myself at all. I need not think, if only I can pay: others will readily undertake the irksome work for me.[Kant, 1784]

Wilhelm von Humboldt would add to the idea:

> Whatever man is inclined to, without the free exercise of his own choice, or whatever only implies instruction and guidance, does not enter into his very being, but still remains alien to his true nature, and is, indeed, effected by him, not so much with human agency, as with the mere exactness of mechanical routine. The ancients, and more especially the Greeks,

> were accustomed to regard every occupation as hurtful and degrading which was immediately connected with the exercise of physical power, or the pursuit of external advantages, and not exclusively confined to the development of the inner man. Hence, many of their philosophers who were most eminent for their philanthropy, approved of slavery; thereby adopting a barbarous and unjust expediency, and agreeing to sacrifice one part of mankind in order to secure to the other the highest force and beauty. But reason and experience combine to expose the error which lies at the root of such a fallacy. There is no pursuit whatever, nothing with which a man can concern himself, that may not give to human nature some worthy and determinate form, and furnish fair means for its ennoblement. The manner of its performance is the only thing to be considered; and we may here lay down the general rule, that a man's pursuits react beneficially on his culture, so long as these, and the energies allied with them, succeed in filling and satisfying the wants of his soul; while their influence is not only less salutary, but even pernicious, when he directs his attention more exclusively to the results to which they conduce, and regards the occupation itself merely as a necessary means. For it is the property of anything which charms us by its own intrinsic worth, to awaken love and esteem, while that which only as a means holds out hopes of ulterior advantage, merely interests us; and the motives of love and esteem tend as directly to ennoble human nature, as those of interest to lower and degrade it. [pp. 20-21, Humboldt, 1792 Printed 1854]

As we have seen in the Introduction, Galileo had the attitude proper of the Enlightenment, rejecting the demand of the Scholastics of supporting his arguments with the writings of previous thinkers and demanding that reason were considered above all teaching.

Scholasticism refers to the philosophy that predominated between the 9th and 17th centuries, but the general attitude that Galileo confronts is not limited to those times. Scholasticism is a form of authoritarianism in which the "Truth" is transmitted in the School from teachers (Masters) to students. In the Enlightened attitude the judging comes from within us and the internal coherence of our thoughts becomes a demand for our unity; reason demands internal coherence on us. It is this demand what reflects on our organisation of the world. Our intellectual unity projects over Nature, we seek to perfect our understanding finding the unity of Nature which corresponds to the unity of Reason. In contrast, for the scholastic spirit, knowledge and understanding of Nature comes from outside, it comes from the teaching of the masters that we, understandably, elevate to authorities. That was the case of Aristotle in Galileo's time.

The mission of the University preparing professionals is one in which the student must raise to master the proven knowledge and to apply it to practical situations with excellent expertise. We certainly do not want physicians testing on us (patients) their last minute conjecture, nor we want engineers building bridges without following proven techniques. The formation of the professional is then inclined towards the scholastic form. Kant termed the faculties engaged in teaching professions the "major faculties" while the faculty of philosophy that aimed at the developing of the human being from inside was called the "minor faculty" [Kant, 1798]. Kant shared with Galileo the idea that reason was the superior authority and correspondingly, the minor faculty had to supervise the major faculties. Such ideas are in the basis of Humboldt's university, which flourished in Germany during a short time, approximately 1810–1870.

How was knowledge first produced? Can understanding be really transmitted? Do ideas degrade when taught? Can ideas recover from years (centuries) of scholasticism?

William Whewell thought on these matters:

> When a theory has been established in its general form, our knowledge of the distribution of its phenomena in time and space can be much promoted by ordinary observers scattered over the earth, and succeeding each other in time, provided they are furnished with instruments and methods of observation, duly constructed on the principles of science; but such observers cannot in any degree supersede the discoverer who is first to establish the theory, and to introduce into the facts a new principle of order. When the laws of nature have been caught sight of, much may be done, even by ordinary observers, in verifying and exactly determining them; but when a real discovery is to be made, this separation of the observer and the theorist is not possible. In those cases, the questioning temper, the busy suggestive mind, is needed at every step, to direct the operating hand or the open gaze. No possible accumulation of facts about mixture and heat, collected in the way of blind trial, could have led to the doctrines of chemistry, or crystallography, or the atomic theory, or voltaic and chemical and magnetic polarity, or physiology, or any other science. Indeed not only is an existing theory requisite to supply the observer with instruments and methods, but without theory he cannot even describe his observations. He says that he mixes an acid and an alkali; but what is an acid? What is an alkali? How does he know them? He classifies crystals according to their forms: but till he has learnt what is distinctive in the form of a crystal, he cannot distinguish a cube from a square prism, even if he had a goniometer and could use it. And the like impossibility hangs over all the other subjects. To report facts for scientific purposes without some aid from theory, is not only useless, but impossible. [p. 80, Whewell, 2016]

We can turn to Piaget as well,

> The basic defect in an empiricist interpretation is that of neglecting the activity of the individual. The entire history of physics, the most advanced of the disciplines founded on experiment, is enough to show that experiment on its own is never sufficient, and that the progress of knowledge is the work of an inseparable union between experiment and deduction. This again suggests a necessary collaboration between the data afforded by the object and the actions or operations of the subject—these actions and operations themselves constituting the logico-mathematical framework outside which the individual never succeeds in intellectually assimilating the objects. Even in sciences as little evolved (in relation to physics) and as purely "empirical" in appearance as zoology and systematic botany, the classificatory (and consequently already logico-mathematical) activity of the individual remains indispensable in order to assure an objective reading of factual data.[p. 745, Gruber and Vonèche, 1995]

We could keep going on citing Kant, Goethe, Peirce and other original people that have thought about it, people that had the audacity necessary to know. The scholar system teaches us to trust the teacher and persuades us that the correct answer is the one that which is in agreement with the lessons of the authority. Truth must be sought outside us. We are trained, as any animal, to adopt this attitude by a system of positive and negative reinforcements [Chance, 1999] such as recognition and promotion or their absence. The result of years of training is usually the abandonment of any behaviour supporting independent thinking. Needless to say, without independent thinking, critical thinking is impossible. Schooling is in the interest of the established power since critical thinking cannot be restricted to a field of expertise, it constitutes a general threat.

The school system tends to dehumanise us, to make us imitators that think with ideas handed down to us. In the terms of Humboldt, it trains us to think with exactness following mechanical routines. In such form, our brain is used as a biological computer, an obsolete kind of computer at the beginning of the XXIst Century. Once we have been dehumanised we can only expect to be replaced by proper machines performing the same tasks far more efficiently. Once we have agreed to call intelligence to this schooled attitude, we can only expect Artificial Intelligence to outperform us.

There are two clearly different goals when studying Newton's physics. One goal would be acquire the abilities necessary to profit from the use of his understanding, the second possible goal is "to be Newton". This is, Newton was an Enlightened person that though by himself, an original thinker. There is no form to be original through imitation of his thoughts but we can imitate him in the sense of developing our capabilities to the maximum power that they can reach given our natural limitations. This is the path of the bildung, the path to Enlightenment. The exercise of reconstructing physics we attempt in this book has been devised as a form of promoting our bildung and at the same time of illustrating how physics depends on our construction.

In the course of our exercise we have encountered too many times the attribution of ideas to old masters that were not really in the writings that they left us. This finding would have not surprised Goethe who wrote:

> You don't have to have seen or experienced everything for yourself; but if you want to trust another person and his descriptions, remember that you are now dealing with three factors: with the matter itself and two subjects.[§570, Goethe, 1832]

If we cannot perceive our intervention in the production of meaning, if we cannot find a hint regarding the difference between observation and fact, we can hardly understand Goethe' saying. This is an ancient problem, Chuang Tzu in Autumn Floods (400 BC) wrote:

> Jo of the North Sea said, "You can't discuss the ocean with a well frog—he's limited by the space he lives in. You can't discuss ice with a summer insect—he's bound to a single season. You can't discuss the Way with a cramped scholar—he's shackled by his doctrines. Now you have come out beyond your banks and borders and have seen the great sea—so you realize your own pettiness. From now on it will be possible to talk to you about the Great Principle. "[Chuang Tzu, 1968]

Any one who is only the product of schooling will have a hard time understanding us. There are some examples that impress us. Several distinguished authors have made important efforts to understand Hegel's logic, for example Popper [Popper, 1963], Carnap [Carnap, 1959] and Bunge [Bunge, 1975] and none of them managed to succeed according to their own accounts. But rather than understanding Hegel, what these philosophers have attempted was to put Hegel's forms of exploring and retrogressing to the truth under their own forms of deductive thinking, without noticing that Hegel had pronounced the analysis of forms usually called logic to be insufficient to reach any truth. Being logic deductive, it cannot retrogress and cannot tell us why, for example, we accept this or that axiom. These authors have renounced to know what is behind the axioms and try to place Hegel in terms of the knowledge they allow to themselves. In Hegel,

> §94. Logic is pure science, that is, pure knowledge in the entire range of its development. But in the said result, this Idea has determined itself to be the certainty which has become truth, the certainty which, on the one hand, no longer has the object over against it but has internalised it, knows it as its own self —— and, on the other hand, has given up the knowledge of itself as of something confronting the object of which it is only the annihilation, has divested itself of this subjectivity and is at one with its self-alienation.[Hegel, 2001a]

In the introduction Hegel explains that the attempt of reducing logic to an analysis of the forms makes it impossible for logic to reach Truth.

> §26 What we indicated as the beginning of the science [of logic] —— a beginning which we have already recognised as having a high value both on its own account and as a condition of genuine knowledge —— namely, the treatment of Notions generally and the moments of the Notion, that is, the determinations of thought, primarily as forms which are distinct from the matter of thought and only attached to it, this attitude directly reveals itself as intrinsically inadequate for the attainment of truth —— and the truth is the declared object of and aim of logic. [...]

The situation is not symmetric, it is not a matter of two different opinions. Hegel does not ignore common logic (Aristotelian in his time) he just declares that he finds it insufficient to reach the truth. In contrast Popper, Carnap and Bunge rule as absurd what cannot be subordinated to their (limited) schemes. Are they just "cramped scholars" in Chuang Tzu terms?

Progress in communication and in sharing has made it possible today to read digitised copies of historic books. What is attributed to an author can be checked and the context reconstructed when needed. This possibility showed us how often the citation distorts the meaning of what is being cited, the original author is censored in those matters that the citing actor does not want to support. The famous words of Galileo with respect to mathematics being the language in which Nature is written (cited in the introduction in its original language) are a selected part of a paragraph in which Galileo exposes his criticism to the scholastic attitude, but the criticism has been censored. The metaphor is then distorted. Galileo intends to transmit the phenomenological method. The Truth about Nature is to be found in Nature itself, not so much in the scholastic commentary, and it is to be ciphered in mathematical relations. The vindication of Reason as the instrument provided to humanity by God to learn about Nature, and the supremacy of Reason when confronting scholarly opinion completes a picture that is not very favourable to the professional scholar since their authority is questioned.

Because of the previous observations we have made a point to provide the context of the phrases cited and the precise location of them so that the readers that rightfully distrust us have their checking facilitated and Truth prevails. Our main goal is to help the reader in their bildung, we facilitate critical reading to help them to become enlightened.

1.A.2. E. Mach's fixed stars and "absolute motion". During the late XIXth Century a controversy emerged between German Physicists. After Thomson's work [Thomson, 1884] introducing "Inertial frames", E. Mach went back to the notion of absolute space in relation to the "fixed stars" [pp. 542-543, Mach, 1919] and was sharply criticised by Lange [1886] and Streintz [1883]. Streintz also indicated his suspicion that Mach had no knowledge of Newton and Euler. In his controversy with Lange and Streintz, Mach writes

> I never assumed that remote masses only, and not near ones, determine the velocity of a body (Streintz, p. 7); I simply spoke of an influence independent of distance.

Mach idea that distant bodies (like the fixed stars) may exert an influence in motion was also criticised (ridiculed?) by Poincaré:

> We must also take care to distinguish between the different kinds of hypotheses. First of all, there are those which are quite natural and necessary. It is difficult not to suppose that the influence of very distant bodies is quite negligible, that small movements obey a linear law, and that effect is a continuous function of its cause. I will say as much for the conditions imposed by symmetry. All these hypotheses affirm, so to speak, the common basis of all the theories of mathematical physics.[p. 169, Poincaré, 1913d]

It is worth to further consider Mach's view because of its social influence. Mach sustains:

> But if we take our stand on the basis of facts, we shall find we have knowledge only of *relative* spaces and motions. *Relatively*, not considering the unknown and neglected medium of space, the motions of the universe are the same whether we adopt the Ptolemaic or the Copernican mode of view. Both views are, indeed, equally *correct*; only the latter is more simple and more *practical*. The universe is not *twice* given, with an earth at rest and an earth in motion; but only *once*, with its *relative* motions, alone determinable. It is accordingly, not permitted us to say how things would be if the earth did not rotate. We may interpret the one case that is given us, in different ways. If, however, we so interpret it that we come into conflict with experience, our interpretation is simply wrong. The principles of mechanics can, indeed, be so conceived, that even for relative rotations centrifugal forces arise.
>
> Newton's experiment with the rotating vessel of water simply informs us, that the relative rotation of the water with respect to the sides of the vessel produces *no* noticeable centrifugal forces, but that such forces *are* produced by its relative rotation with respect to the mass of the earth and the other celestial bodies. No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick. The one experiment only lies before us, and our business is, to bring it into accord with the other facts known to us, and not with the arbitrary fictions of our imagination.[p. 232, Mach, 1919]

One must object Mach on the grounds that the motion of bodies relative to the distant stars cannot be observed, much less measured, the concept is a fantasy that makes absolute space out of relational space by providing an imaginary (not practical) absolute and universal reference. Apparently, Mach lacks a way to check whether an idea is rational or not. For example, his fantasy is not rational for it involves something that —as he indicates— cannot be subject to experimental scrutiny. The Ptolemaic system can only appeal those that believe that God made humans using himself as a model, and correspondingly the Earth is the centre of the Creation. Yet, all this is religion and cannot be contrasted with observations. Mach's fantasy goes directly against the No Arbitrariness Principle. Mach rejects reason and admits his own fantasy at the same time that he blames the use of reason as imagination (fantasy). Mach's argumentation is correct in the sense that if Science is merely a matter of social economy [The Economical nature of physical inquiry, Mach, 2012][26] the Ptolemaic and Copernican systems differ only by convenience for there is no Truth in physics, just economic convenience. Once reason is rejected, only experiments and observations can reject a theory, hence fantasies flourish where experiments and observations are difficult, scarce or impossible. If we apply W. von Humboldt's "the motives of love and esteem tend as directly to ennoble human nature, as those of interest to lower and degrade it", Mach's science degrade us. The contrast makes evident the change in the system of values operated during the XIXth Century.

1.A.3. E. Mach and "inertial mass". Another socially successful idea of Mach is the distinction between inertial mass and gravitational mass [pp. 216-222, Mach, 1919]. We observe that the concept of mass is derived from density and volume but density is calculated associated to gravitational forces and quantified with Galileo's balance. To get the density of a body with the balance, the body is first weighted, W, and weighted again while it is submerged in liquid, W', and the volume displaced, V, is measured. Finally the weight W_l of an equivalent volume of liquid is also determined. Using Archimedes' principle and after little algebra we can write:

$$\left(\frac{W'}{W_l}\right) = \left(\frac{\rho}{\rho_l}\right) - 1 = \frac{W - W_l}{W_l}$$

where ρ stands for the density of the body and ρ_l for the density of the liquid. The mass is then, by definition, $m = V\rho$, and we can call it "gravitational mass" because it has been obtained using gravitational forces. The relation $\left(\frac{W'}{W_l}\right) = \frac{W-W_l}{W_l}$ involves only measured quantities and is implied by the assumption of independence of the intervening causative actions (see Definition 3.2). The hypothesis can then be put to test.

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[26] It is the object of science to replace, or *save*, experiences by the reproduction and anticipation of facts in thought. Memory is handier than experience and often answers the same purpose. This economical office of science, which fills its whole life, is apparent at first glance; and with its full recognition all mysticism in science disappears.

Science is communicated by instruction, in order man may profit by the experience of another and be spared the trouble of accumulating it for himself and thus, to spare posterity, the experiences of whole generations are stored up in libraries. [p. 481, Mach, 1919]

Let us now gauge the force of a spring at various elongations, Δ_l, we fix one end of the spring to the ground and attach a string to the other end. The string goes trough a pulley and at the other end of the string we hang different masses previously weighted. In each case we wait until all motion ceases and build a table that relates elongation of the spring to weight of the mass. This is a function

$$m_B g = W = f(\Delta_l)$$

that calibrates the spring (we have introduced m_B which is the gravitational mass of the body used to make each comparison, thus we can write $l := l(m_B)$ as well.

Assume now that the relation between acceleration and force is: $m'a = F$ with $m' \neq m$, and call m' the "inertial mass" as Mach did. There cannot be any prescription regarding which source of force we should use because such prescription would make the concept of force inconsistent. For example, the force provided by a spring F_S that balances a weight W, would provide an acceleration to a body which is different than the acceleration provided by the weight, in formulae $m' \equiv \frac{F_S}{a_S} \neq \frac{W}{a_W} = m$, where the last equality is due to the construction of the concept of mass and force. Since the spring balances the weight we write $F_S - W = 0$. However since $a_S \neq a_W$ according to the proposed relation there would be a remaining acceleration $a_S - a_W \neq 0$ and the spring would not be balanced by the weight. A contradiction that means that either Newton's formulae are incorrect or the assumption $m' \neq m$ is incorrect.

If we accept, perhaps as idealisation, that the sum of causes of motion over a body produces an acceleration which is equal to the sum of the accelerations produced by each cause, we are entitled to test the relation $m'a = F$ using gravitational forces, but in such a case, we know by construction that $m = m'$. The equality of the inertial mass and the gravitational mass needs not to be checked experimentally, the possibility of them being different is null in as much as we understand the construction of the meaning of force and mass. Mass, weight, force and Newton's second Law emerge together and the terms make sense only in relation to each other. They do not preexist Newton's work.

In contrast, if we let force just to be a name for causes of acceleration and allow us to forget that they are generalisations of the intuited concept of weight we leave forces disconnected from the complete intuition associated to them and to the fact that we gauge forces against weight. We believe that this is the result of a teaching that introduces Newton's equations without connecting then to the primary intuitions that are part of their legitimisation. Such procedure turns Newton's equations somewhat metaphysical since the connection with experience and previous knowledge is damaged. Unfortunately, Newton's presentation in terms of Axioms leaves us without clues about how he developed the theory.

1.A.4. Criticism of Mach's approach to "inertial" mass. Let us examine Mach criticism to his understanding of Newton. Notice that it is not possible to separate his intervention reading Newton from Newton's ideas, what Mach criticises is what he understood in Newton or in the sources he consulted. Mach begins [p. 216, Item 1, Mach, 1919] by declaring that mass and the third axiom in Newton cannot be separated. No reason offered so far. Next (Item 2) he declares that the expression “quantity of matter” cannot be accepted for mass because it lacks the requisite of clearness. It is a judgment for which he offers no other support than his ego (I know it) and leaves behind that Newton based the concept of mass in the concept of density and volume which are measurable. He continues in the same item

> When therefore, with Newton, we make the assumptions, respecting pressure due to weight, that $p = mg$ and $p' = m'g$, and put in conformity with such assumptions $p/p' = m/m'$ we have made actual use in the operation thus performed of the *supposition*, yet to be justified, that different bodies are measurable by the *same* standard.

Clearly, this is not Newton’s construction but Mach’s. The concept of mass is devoid of its connection with static, with Archimedes’ principle. It makes of mass a purely kinematic quantity. In Item 3 (same reference, p. 217) he proceeds to compare two bodies in the (impossible) case that they are identical:

> 3. When two bodies (Fig. 140a), perfectly equal in all respects, are placed opposite each other, we expect, agreeably to the principle of symmetry, that they will produce in each other in the direction of their line of junction equal and opposite accelerations. But if these bodies exhibit any difference, however slight, of form, of chemical constitution, or are in any other respects different, the principle of symmetry forsakes us, *unless we assume or know beforehand* that sameness of form or sameness of chemical constitution, or whatever else the thing in question may be, is not determinative. [...]

It is Mach, not Newton who is trying to reach the concept of mass through the intervention of Newton’s third law. Newton states in his first definition

> The quantity of matter is the measure of the same, arising from its density and volume conjunctly.

He next indicates that he means (in modern notation) $m = V\rho$, where V stands for the volume and ρ for the density. It is our impression that what Mach does not understand is that “quantity of matter” is not something that carries meaning by itself, it is a name that Newton gives to the intuition that is made sensibly manifest on the form proposed. Thus, quantity of matter is that thing that is revealed by the experience of our senses and measured as indicated. The definition points to things that we know by our senses, not things that live in our minds. It is the map of the observed phenomena into mathematics. It requires personal experience since the expressions in a book or the class by the professor only produce evocations, links to other ideas be them real or imagined. The repetition during a few generations of the instruction cycle advocated by Mach (see footnote 26) might result in a severe hampering of the phenomenological capabilities of scientists.

How can Mach think that his reading is universal and that the substitutions he makes in Newton’s thoughts do not change the meaning of what Newton thought?

Detaching concept from construction is a sort of criminal act for a philosopher. Hegel put it bluntly:

> *Note.*—Philosophy has to do with ideas or realized thoughts, and hence not with what we have been accustomed to call mere conceptions. It has indeed to exhibit the onesidedness and untruth of these mere conceptions, and to show that, while that which commonly bears the name "conception," is only an abstract product of the understanding, the true conception alone has reality and gives this reality to itself. Everything, other than the reality which is established by the conception, is transient surface existence, external accident, opinion, appearance void of essence, untruth, delusion, and so forth. Through the actual shape, which it takes upon itself in actuality, is the conception itself understood. This shape is the other essential element of the idea, and is to be distinguished from the form, which exists only as conception.
>
> *Addition.*—The conception and its existence are two sides, distinct yet united, like soul and body. The body is the same life as the soul, and yet the two can be named independently. A soul without a body would not be a living thing, and vice versa. Thus the visible existence of the conception is its body, just as the body obeys the soul which produced it. Seeds contain the tree and its whole power, though they are not the tree itself; the tree corresponds accurately to the simple structure of the seed. If the body does not correspond to the soul, it is defective. The unity of visible existence and conception, of body and soul, is the idea. It is not a mere harmony of the two, but their complete interpenetration. There lives nothing, which is not in some way idea. The idea of right is freedom, which, if it is to be apprehended truly, must be known both in its conception and in the embodiment of the conception. [p. 21, Hegel, 2001b]

What Mach is trying to do is to give meaning to void concepts he has received, he is producing a reconceptualisation that is in agreement with his own reasoning. He certainly cannot do otherwise unless he surrenders his freedom to some power or authority. He finds then that the conceptualisation he produced to accept Newton's law is not good enough for him and proceeds to amend it. There would be no objection to his writings had he written not "Criticism of the principle of reaction and of the concept of mass" but rather "Criticism of my understanding of the principle of reaction and of the concept of mass". In the same form, we criticise what we understand of Mach but contrary to Mach we indicate the reasoning behind our criticism and do not substitute his sayings by our hearings.

The difference is the self-consciousness of our conscience. There is a transcendent step in acquiring self-consciousness. While our ordinary conscious efforts make us aware of the environment (the universe minus the conscious being), self-consciousness requires a bit more, namely being aware of our awareness. Self-consciousness requires a master consciousness that controls the doings of the self-conscious being that is operating the knowing. There is a long and fundamental discussion regarding this matter in [Hegel, 1966]. In a language not available to Hegel we can say that self-consciousness implies the fractalisation of our consciousness, for in a fractal structure the set A contains a (proper) subset equivalent to the whole set: $A \supseteq \left\{B, \widetilde{A}\right\}$ and $\widetilde{A} \sim A$. In the present context, B represents our consciousness of other things in the universe and $\tilde{A} = A$. The question: why am I persuaded by some argument? triggers a reflection going "backwards" from the conclusions to the foundations that is present in every step of our reasoning (cf. [p. 29, Hegel, 2001a]). The question: why am I persuaded by some argument? leads to a potentially infinite regression in which the understanding achieved by the master consciousness becomes in the act a matter that needs to be understood in the self-consciousness.

1.A.5. Criticism of absolute space, and absolute motion. The association of Newton with absolute space is also well known but the distinction he made between absolute space and true space is almost never discussed despite it being fundamental to understand Newton's laws of motion. While True motion can be distinguished by our senses and actual experiments can be at least proposed to check it, attempts to make sense of absolute space appear invariably linked to imagination. DiSalle [DiSalle, 1993] in his study of Carl Neumann explains Neumann's proposal of considering an arbitrary Alpha-body, just a reference frame whose existence is postulated in the same form that in his time the ether was postulated. Unlike the ether, Neumann's Alpha-body was ridiculed. The fixed stars play the same role than the Alpha-body but for reasons beyond our understanding they have been more convincing, and the same can be said of some "centre of mass of the universe" as inaccessible as the Alpha-body. In contrast, Kant's perception of space being a necessity of our form of thinking and the support that Piaget's studies provide to the idea continue to be rejected. Recently, Solomon [2023] discussed Newton's example of the two globes [p. 82 Newton, 1687]. The paper discusses the history of the diverse readings given to Newton's example. The classical interpretation:

> [...] on this reading, the example of the globes is lumped together with the example of a rotating water bucket. Their joint role is to show the existence of absolute motion (and by inference to the best explanation, the existence of absolute space). The tension in the cord shows the endeavor to recede from the center. The existence of the endeavor to recede from the center signifies in turn the existence of real motion. Such motion is not motion with respect to any body, since it is implicitly assumed that there are no other bodies in the universe and there is no change in relative distance between themselves. Therefore, this is absolute motion. Some authors would go further and clarify the implicit inference: because

> absolute motion exists and since absolute motion is motion with respect to absolute space, then absolute space also exists.

A new interpretation was put forward, in part due to Rynasiewicz [1995b,a]. In this new reading "we *assume* that True motion and absolute motion coincide; we no longer seek to prove either the existence of absolute motion, or that a body's True motion should be defined as motion with respect to absolute space." [Solomon, 2023]. In both interpretations dominates the perception that we are in front of a thought experiment, i.e., a fantasy that can only take place in our minds. However, it is easy to propose an actual experiment testing the idea[27], or, as we have done with the hammer throw, an equivalent experience. How can True motion and absolute motion be equated when Newton says that absolute motion cannot be determined, but there is a unique true motion?

Again, the problem of scholars is that they interpret their readings in terms of their form of understanding (which may include fantasies), and unless the fantasies are kept checked by experience they plunge without knowing into metaphysics.

1.A.6. On action at distance. Another frequent source of misreading of Newton's mechanics is our primitive intuition on how influences are exerted. In her discussion of action at distance Hesse [1955] distinguishes three forms of explanations:

> Fundamental actions between parts of matter and aether were conceived in three different ways: as impacts, as actions in a continuous medium, and as actions at a distance.

Impacts and actions in a continuous media come from familiar experience. In contrast, action at distance does not, unless we play with magnets often. In our primitive intuition, matter refers to that impenetrable thing that we recognise by tact and vision. The first two methods are unified in the motto "Matter cannot act where it is not". When we have to consider gravitation or the interactions of magnets we face a dilemma: either we protect our simple intuition and conjecture the existence of a material placed where we perceive nothing, or perhaps we imagine some sort of invisible particle being sent from one body to the other[28]; the second option is to restrain us of providing explanatory theories which cannot be

[27] Take two masses of equal weight, form, material,... and attach to each of them a steel rod of the same length in identical form. Connect each rod to one end of a spring and have the spring and rods limited in motion by a transparent tube that surrounds them so that they can only move apart (close up) by pulling (pushing) on the spring although each one can rotate around the other (notice! rotation does not need to be specified in front of something external as in the standard interpretation). At the middle extension of the tube we attach a rod providing a mechanism to spin the tube, the balls and their connection on the plane perpendicular to the direction of gravity. Perhaps with the aid of a stroboscopic light you will find out that as the angular velocity is increased the two balls move apart increasing the elongation of the spring. The same experiment can be performed without the spring, but we recommend to exercise caution since the force of the spring is essential to the result.

[28] If the invisible particle and the invisible ether have no other effect than "explaining" the observations that motivated their introduction, they respond only to a psychological necessity. They are not scientific and much less real, yet, if it helps to produce new ideas through fantasies it may serve a practical function. In any case, we must not forget they are fantasy, not reality.

confronted with experience. The latter attitude in front of the unknown phenomena correspond with the *epojé*[29]. Action at distance is the result of the *epojé*, we admit what we observe (for example that there is no contact between the matter of the magnets but yet they can pull or push each other despite being distant) and proceed with what we know without introducing inventions that are required only to protect our prejudices. "I frame no hypotheses" [p. 506, Newton, 1687] is the form that the *epojé* takes in Newton.

1.B. Scholium. A changing epistemic praxis

1.B.1. Newton's rules of reasoning in experimental philosophy. Newton wrote in the Principia his laws of thought [pp. 384-385, Newton, 1687], a material seldom commented. These laws are part of his epistemological legate:

> Rule 1. We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances. ("Nature does nothing in vain" and "Nature is pleased with simplicity, and affects not the pomp of superfluous causes".)
>
> Rule 2. Therefore, to the same natural effects we must, as far as possible, assign the same causes
>
> Rule 3: The qualities of bodies, which admit neither intension nor remission of degrees, and which are found to belong to all bodies within the reach of our experiments, are to be esteemed the universal qualities of all bodies whatsoever.

"We are certainly not to relinquish the evidence of experiments for the sake of dreams and vain fictions of our own devising nor are we to recede from the analogy of Nature, which uses to be simple, and always consonant to itself."

> Rule 4: In experimental philosophy we are to look upon propositions collected by general induction from, phenomena as accurately or very nearly true, notwithstanding any contrary hypotheses that may be imagined, till such time as other phenomena occur, by which they may either be made more accurate, or liable to exceptions.

"This rule we must follow, that the argument of induction may not be evaded by hypotheses"

The rules constitute a sort of epistemic hygiene. The first two correspond to the No arbitrariness principle. Newton does not explain what are "true causes", however, in his discussions of, for example, impenetrability of matter, he considers the source of truth what comes from the senses. What we will later call "the observed real" is what must be considered true. The fourth law confirms the same criteria requesting to drop hypotheses contrary to observations and to what is inferred from such observations ("collected by general induction") at least until

[29]Variously spelled: epojé, epoché and epokhé from Greek ἐποχή, suspension or bracketing of the judgment [§ 33, Husserl, 1983].

new phenomena cast light on them. The third rule deserves a long explanation in Newton. He explains:

> That abundance of bodies are hard, we learn by experience; and because the hardness of the whole arises from the hardness of the parts, we therefore justly infer the hardness of the undivided particles not only of the bodies we feel but of all others. That all bodies are impenetrable, we gather not from reason, but from sensation.

Once again we see the NAP in action. What is granted as a property to the bodies that are accessible to us cannot be denied to those bodies not accessible to experimentation. For example, if the Earth attracts bodies and make them fall onto its surface, we cannot deny such property to other planets or to the Moon. Actually, the attraction by the moon of the seas can be considered a confirmation of our inference.

1.B.2. Is the epistemic praxis a matter of psychology? Long time before we came across Galileo's saying regarding the Truth being inside us (footnote 2), long before we read Chuang Tzu's *Autumn Floods* or the quotation that Ortega y Gasset makes of it [p. 5, Ortega y Gasset, 2004-2010] we were trying to understand why very well prepared academicians had difficulties with Hegel's dialectic, Husserl's phenomenology and Peirce writings. For us, with greater degree of elaboration by these authors, their ideas evolved naturally from the questions and observations we had. Yet, professional academics in philosophy had difficulties with them, except perhaps the specialists and certainly Alejandro Guillermo Romero Fernández.

Eventually, it became obvious that there was an affinity in our thinking with some authors while other authors repelled us despite that we were able to follow them. These latter authors were the core reference in epistemology. Some other authors such as Kant and Newton appear to be appreciated by others as much as by us. Yet, in closer inspection, what we read and what other people read is different (this is: what the senses receive may be the same but what we learn is different), as it has been shown in the case of Newton.

It became evident that there are two extremes in learning that oppose each other. What is evident is the opposition and it is the opposition what reveals the opponents, the opponents do no exist before we open this window to the observed-real making it an ideated-real as well, and together part of reality. The opponents do not preexist the opposition they are constituted by idealisation, as limiting cases of what it is observed as being in opposition. We call this cognitive operation *dialectical opening*. It became a source of happiness to see that Piaget's experimental findings seem to reflect the same process. Much later, intelligence has been characterised by the opposition of two modes: crystallised and fluid [Cattell, 1963, Horn and Cattell, 1967, Sanginabadi, 2020], the fluid intelligence predominating in the early years of life, the crystallised being characteristic of adulthood. Sanginabadi [2020] resumes:

> [...] fluid intelligence (Gf) as "the ability to reason and to solve new problems independently of previously acquired knowledge". Fluid intelligence (Gf), which is necessary for all sorts of logical problem-solving tasks, includes both inductive and deductive reasoning. Gf is the capability of a person in understanding patterns and relationships and using logic and abstract reasoning to analyze and solve novel problems.
>
> [...] Crystallized intelligence, however, is the ability to apply experience, knowledge, and skills in solving new problems, and it relies on the information in the long-term memory. Crystallized intelligence (Gc) indicates the lifelong acquisition of knowledge through education, language, and culture, and the ability of thinking and reasoning using words and numbers.

The fluid intelligence constructs the structures for understanding. In contrast, the crystallised intelligence has rigid structures resulting from previous understanding, habits and most often instruction, thus it would be apparent that the fluid intelligence is more apt to construct original theories while the crystallised intelligence is more apt to enlarge and exploit existing theories. Instruction is the base of the latter while the former is innate although it improves with schooling in children [Stelzl et al., 1995]. A second psychological aspect worth mentioning is the post-formal stage of development proposed by some authors (see for example [Kramer, 1983]). The stage is characterised by:

> [...] three shared features, (1) the realization of relativistic non-absolute nature of knowledge; (2) an acceptance of contradiction; and (3) integration of contradiction in an overriding whole.

According to the proponents it is the only stage that occurs during adulthood and it is the time when duality and dialectical thinking develop. Not all subjects progress to this last stage. In the formal stage the analytical judgment is developed, the post-formal period develops the synthetic judgment.

We believe that the writings of pioneers in science show elements that correspond to a persistent fluid intelligence and to post-formal thinking. In contrast, the logicians appear to excel in their use of the formal intelligence. We notice that Pepper [Pepper, 1942] in his study on evidence recognises four "pure" styles of constructing world hypotheses and mentions philosophers in which one form prevails noticeably over the others (Ch. VII. A general view of hypothesis) the classification emerges from two discriminating oppositions: analytic vs synthetic and dispersive vs integrative. The post-formal stage is characterised by synthetic judgements [Kramer, 1983].

Concerning the opening question of this Subsection, there are certainly psychological factors influencing the natural philosopher's approach to knowledge. For example, understanding takes a different shape in the post-formal stage as compared with the formal stage. In Peirce's terms, we may say that the conditions for the cessation of doubt and acceptance of a belief are not the same for all producers of science. There is however a sort of asymmetric inclusiveness in understanding since while the post-formal or dialectic thinker can deal with formal or logic thought, the converse seems to be much harder.

Alongside with the use of analogy emerges the frequent use of phantasy of which Einstein's "*Gedankenexperiment*" is an example, as well as Hertz' "*bild*" and Mach's advocacy of the use of memory in place of observations as "economy of thought".

> For most people, visual imagery is an innate feature of many of our internal experiences, and appears to play a critical role in supporting core cognitive processes. Some individuals, however, lack the ability to voluntarily generate visual imagery altogether – a condition termed "aphantasia". [Dawes et al., 2020]

Aphantasia is not a disorder but rather a neurodivergency, its prevalence in an studied population of USA was of 9% (self-reporting) and 1.5% (questionnaire). This means that a large majority of students are expected to find their acceptation of instructed theory facilitated by the use of self generated sensory imagery (*phantasy*). It is then not surprising that when the instruction of physicists became massive, the cognitive form named "*Bild*" in German and portrayed by Hertz in his mechanics [Hertz and Walley, 1899] emerged alongside with the ether. The latter was precisely the form of accepting Maxwell's equations advocated by Hertz.

> [...] for through his model Maxwell reached those equations whose peculiar and almost inconceivably fantastic powers were described so vividly by the man best qualified to do so, namely Heinrich Hertz (in his lecture on the relation between light and electricity, published in Bonn, 1890). To this I wish to add only that Maxwell's formulae were merely consequences of his mechanical models, so that Hertz's enthusiastic praise is due in the first place not to Maxwell's analysis but to his ingenuity in discovering mechanical analogies [p. 10, Boltzmann, 1974].

But perhaps Hertz had unmatched, extraordinary, imagination (phantasy), called hyperphantasia and accounting for the 1.5% of the population as well [Zeman, 2024]. Boltzmann informs us:

> [...] I have often heard Hertz's mechanics praised yet never seen anybody pursue the path he indicated [p. 88, Boltzmann, 1974].

Phantasy is supported in memory, very much as it is the crystallised intelligence that predominates (statistically) in adulthood. Thus, instruction in physics may be selecting a psychological profile defined by good memory, crystallised intelligence and formal thought, a profile justifiable in terms of "efficiency" although it leaves aside the aphantasy lot, despite that the challenges that these students face make them more creative when able to overcome the recommended use of imagination. In particular, they are stronger in *divergent thinking* [Maw et al., 2024]:

> Divergent thinking, as we have already seen, involves producing multiple answers through processes like shifting perspective on existing information (seeing it in a new way) or transforming it, for instance, through unexpected combinations of elements usually not regarded as belonging together. The answers arrived at via divergent thinking may never have existed before. Sometimes this is true merely in the experience of the particular person or the particular setting, but it may involve what Boden (1995) called "radical originality".[Cropley, 2015]

Contrasting with divergent thinking, there is convergent thinking, known as deductive thinking as well, completing the parallelism with fluid and crystallised intelligence. The current research in psychology casts light over the emphasis of logicians like Mach and highly acclaimed scientists like Einstein that negated the existence of forms of reasoning different from deduction.

We will presently discuss our contribution to understanding in the form of the No arbitrariness principle, already advanced in Section 1.1.

1.B.3. No arbitrariness principle (NAP). According to Ballard [1960], Leibniz principle of sufficient reason (PSR) made little or no progress since the Clarke-Leibniz polemic. A discussion among substantivalists and relationalists reemerged with relativity theory but in a form in which philosophy is subordinated to the "success" of physics. Within such an intellectual disposition it has been declared that "neither the PSR nor the Principle of Identity of the Indiscernibles (PII) "enjoys at present unquestionable philosophical credentials" [Maudlin, 1993]. Actually, the discussion appears as frozen in time, lacking critical contributions and it still revolves around God.

In contrast, NAP stands for a critical as well as operative view, it has a long tradition in mathematics where the expression "without loss of generality we can assume..." is frequent in proofs, meaning that the result will not depend on our arbitrary choices but for the sake of the argument a choice must be made. This is: arbitrariness should not and will not bear any consequence in what is being proved. The metacognitive instruction received by every student of physics: "the final result should not depend in your election of units or your choice of path to reach it" manifests the same conviction: arbitrariness has no part in truth.

Mach refers to a principle of sufficient reason in relation to Archimedes [p. 9, Mach, 1919] and the equilibrium of levers and in several instances as "principle of symmetry" and "Archimedes' principle of symmetries", but his notion appears to be confined to mechanics. Barbour [Barbour, 1982, 2010] has worked on formalising Mach's principle in relation to space-time and special relativity. In as much as he worked in the context of Mach's mechanics he does not address Mach's misunderstandings. The No Arbitrariness Principle is not a principle of mechanics but a principle of reason, as such, its empirical base is much larger than physics, as it is the empirical basis of logic and larger than those, the basis of reason. The rejection of arbitrariness can be found in Kant's first formulation of the *categorical imperative:* "Act only according to that maxim whereby you can, at the same time, will that it should become a universal law" [4:421, Kant, 1993]. It can be found as well in law, as for example the declaration of human rights by the French Assembly of 1789 "Men are born and remain free and equal in rights. Social distinctions may be based only on considerations of the common good."[30] It is the internal urge of reason what demands equality.

In our attempts to understand Nature as it is, we recognise that any knowing we acquire by observation is contaminated by the observer. To remove the intervention of the subject it is required for other subjects to observe the phenomena contaminated by their own subjectivity. All subjects have then their own perspective and we ought to accept that they are valid, and that if we were allowed to substitute ourselves as observers with all other circumstances kept unchanged we would have the same perspective than the replaced observer, which is to say that reality is one, observers are equal in principle and all what can change are the circumstances of the observations. We certainly know that this is a good form of thinking for after all, considering the toy I have in my hands now to be the same toy with which I played yesterday in manners so enjoyable allows me to repeat yesterday's actions, repeating with them the enjoyment. The rule appears as an adaptation to life, but it applies not to identical situations as Mach purports, it applies to an object that has been stripped of its circumstances, be them the spatial relations to other objects or the simultaneity with other events (time) and much more. It might very well be that a ball remains being a ball in as much as it bounces off the ground when we let it fall. The day it stops bouncing we consider it not longer a ball (and almost certainly cry). Thus, the principle applies not only to perfect, mathematical, equivalences but rather, it applies to the real whose relevant determinations have been previously selected creating an abstract object, an idea that relates to the material, sensorial, object.

The use that we have given to NAP in this chapter transcends the application to formal mathematical structures such as equations of motion or a mathematised space (which is the realm where the application of crystallised intelligence to physical problems leads us). We are not using formal operations but rather post-formal, synthetic thinking. In the next chapter we will schematise our view of reality.

[30]The declaration of the rights of man and the citizen (French presidency)accessed January 6 2025.

CHAPTER 2

Science, dualities and the phenomenological map

2.1. Introduction

Newton's work became a model for Scientific Research, especially in physics, but the form in which the model was to be understood differs among those that agree on the exemplary character of it. Once again, the same objective matter becomes different (and presents opposing) perspectives depending on the subject that enunciates them. Two groups are clearly distinguishable and both have illustrious members. The first group works by analogies, basically its doubting stops when a mechanical analogy of the new phenomena is found to be satisfactory. This group will latter evolve from mechanical analogy into analogy of the forms or formal analogy. The second form for the model is related to Newton's rules for experimental philosophy (see subsection 1.B.1) and is less numerous today but was the leading group during the Enlightenment. Names like Galileo , Isaac Newton, Michael Faraday and Carl Friedrich Gauss belong to this group. Instead of working at the "horizontal" level of analogy this group works by old-fashioned abstraction. The point in which new phenomena like electromagnetic observations is to be related to mechanics is enacted by a cognitive surpass that shows them as different instances of the same form of reasoning. This latter form of proceeding is clearly the "natural" form for those that have made the transition to post-formal thought, while analogy is more "natural" for formal thinkers, and as such it is a strategy more apt for massive instruction (a didactic facilitation in such a case).

Since our interest is to develop the view of physics based upon the philosophy of the Enlightenment we will expose these ideas in the present chapter and will leave for the next chapter a comparison with the method based in analogy.

We will develop how the understanding of Nature has its grounds in a duality, namely the joined participation of sensations and ideas. Mature science organises the duality in a scheme which we will name the *phenomenological map*, where the fundamental guideline is compliance to reason.

2.2. An old philosophical tradition

Ever since the Greeks attempted to conceive an understanding of the world, a duality –a relation of two worlds– has been in the centre of the scene. In various

Socratic dialogues (e.g., [Plato, 2014]), Plato refers to the world of forms, or ideas, an eternal world of perfection, as well as to the world of imperfect copies, our material world. Plato's theory of forms is well known and has deserved extensive discussion[31]. We will simply observe that the duality it introduces has been a substantial part of epistemology since then. We shall call the world of forms *Ideal World* (IW) and the material world, the world accessible with our senses, *Sensorial World* (SW). We credit Galileo for being one of the first in advancing that the IW was populated by a mental operation he called *idealisation* that produced perfect[32] (or at least perfected) models of the observable [Galilei, 1914]. Several authors have worked along this conception. One of the most remarkable has been Husserl [1983] who used the term *ideation* to indicate the process by which the observable was incorporated in our perception as ideas. Piaget and García [1989] made a clear distinction between the observed —that what reaches our senses— and the *facts*, the ideated, that what is incorporated in our knowledge as perceived. All these authors have in common not only the duality between both worlds, but the existence of correspondences between elements in one and the other world. Plato emphasised the relation $IW \overset{\Gamma}{\longmapsto} SW$ while Galileo stressed the inverse relation $SW \overset{\Pi}{\longmapsto} IW$. We shall call the pair (Π, Γ) the *phenomenological map*. As far as we know, the properties and the consequences regarding the assumed existence of a phenomenological map have received little attention in the past. The matter was considered in [Margenau and Mould, 1957] and [Dingle, 1960a], who referred to the phenomenological map as "rules of correspondence" but they did not advance into the implied logical structure. In turn, Feigl [1970] reminds us in his analysis of the "orthodox view of theories":

> In the picturesque but illuminating elucidations used, e.g., by Schlick, Carnap, Hempel, and Margenau, the "pure calculus," i.e., the uninterpreted postulate system, "floats" or "hovers" freely above the plane of empirical facts. It is only through the "connecting links," i.e., the "coordinative definitions" (Reichenbach's terms, roughly synonymous with the "correspondence rules" of Margenau and Carnap, or the "epistemic correlations" of Northrop, and only related to but not strictly identical with Bridgman's "operational definitions"), that the postulate system acquires empirical meaning.

and proposes a more strict correspondence, that he names "bridge laws":

> Let me emphasize once more that this manner of regarding theories is a matter of highly artificial reconstruction. It does not in the least reflect the way in which theories originate. Correspondence rules thus understood differ from bridge laws in that the latter make empirical assertions.

[31] See e.g., Stanford Encyclopedia of Philosophy(accessed 2020-05-04).

[32] Perfection must be understood here in the sense of being faultless and capturing/creating the essence of the World. Ultimately, the two worlds can only be understood as one in front of the other, reciprocally defining themselves as in all the fundamental dialectic relations.

The view of Science that we present in this chapter is strongly influenced by C. Peirce original work. By the end of the XIXth Century Science had moved from Kant's second faculty, the faculty of philosophy (whose concern was reason, always according to Kant), to the first faculty where all the studies of interest for the society at large and in particular for the State belonged. In other terms, Science and especially physics became a profession. Helmholtz, in his discourse: "On academic freedom in German Universities" (1877) [Helmholtz, 1908] testifies about the changes in German Universities and the departure with respect to the concept of University proposed by Humboldt aiming towards human development. Experimental philosophers were coming into (almost) extinction and the new social actor, the professional scientist, was replacing them in the development of science. To our knowledge, the last philosophers that tried to understand the philosopher's science were Charles Peirce (1839–1914) and William Whewell (1794–1866) who introduced the words "Scientist" and "Physicist" to distinguish the new social character from the old experimental philosopher [Yeo, 1993] such as his friend Faraday.

Since in a good degree the evolution of physics in the XIXth and early XXth Century was centred in German physics and its influence, it is important to notice that the social situation of science in the German states presents an important evolution along this time. Lenoir [1998] recognises three periods: 1810-1848, 1848-1871, 1871-1910. The first period starts with the creation of Berlin's University, very much influenced by W. von Humboldt. The free University became the new model to imitate. The duty of professors was to teach but they had ample time to spare around their interests and it was expected (but not subject to contract) they would be involved in research [Ben-David, 1971]. During this time idealistic philosophy flourished and abstract, mathematical, inductive physics developed. Most of the work was performed in individual form, as opposed to laboratories with a head scientist and disciples. By 1840, Justus Liebig, a professor of Chemistry, proposed a new program which "combined pursuit of pure knowledge, typical of science academies, with work appropriate to technical institutes, which trained students in material production" [Lenoir, 1998]. Simultaneously, accompanying Germany's unification process, science began to be perceived as fulfilling two goals. Rudolf Virchow, a scientist and politician, expressed that the "meaning of the natural sciences" lay "essentially in the material benefit that it produces, in the use that it creates" (quoted in [p. 39, Cahan, 1985]) and the second task was to congregate German people in one unit, to achieve a "real unification of minds, to put the many members of the nation on a common intellectual basis where each can then really feel himself as one" (quoted in [p. 40, Cahan, 1985]). The economic goal lead to the creation of research institutes and to specialisation. This is the context where disciplines emerged. The new focused science however brought a decline in the quality of human formation, decreasing the self-critical capabilities [On the Relation of Natural Science to General Science Helmholtz, 1873]. Finally, about 1870 a new change took place and technoscience appeared in the scene [p. 126, Ben-David, 1971].

2.3. A pragmaticist view of the philosopher's science

If we endeavor to form our conceptions upon history and life, we remark three classes of men. The first consists of those for whom the chief thing is the qualities of feelings. These men create art. The second consists of the practical men, who carry on the business of the world. They respect nothing but power, and respect power only so far as it [is] exercized. The third class consists of men to whom nothing seems great but reason. If force interests them, it is not in its exertion, but in that it has a reason and a law. For men of the first class, nature is a picture; for men of the second class, it is an opportunity; for men of the third class, it is a cosmos, so admirable, that to penetrate to its ways seems to them the only thing that makes life worth living. These are the men whom we see possessed by a passion to learn, just as other men have a passion to teach and to disseminate their influence. If they do not give themselves over completely to their passion to learn, it is because they exercise self-control. Those are the natural scientific men; and they are the only men that have any real success in scientific research [CP 1.43, Peirce, 1994].

If we are to define science, not in the sense of stuffing it into an artificial pigeon-hole where it may be found again by some insignificant mark, but in the sense of characterizing it as a living historic entity, we must conceive it as that about which such men as I have described busy themselves. As such, it does not consist so much in knowing, nor even in "organized knowledge," as it does in diligent inquiry into truth for truth's sake, without any sort of axe to grind, nor for the sake of the delight of contemplating it, but from an impulse to penetrate into the reason of things. This is the sense in which this book is entitled a History of Science. Science and philosophy seem to have been changed in their cradles. For it is not knowing, but the love of learning, that characterizes the scientific man; while the "philosopher" is a man with a system which he thinks embodies all that is best worth knowing. If a man burns to learn and sets himself to comparing his ideas with experimental results in order that he may correct those ideas, every scientific man will recognize him as a brother, no matter how small his knowledge may be [CP 1.44, Peirce, 1994].

But if a man occupies himself with investigating the truth of some question for some ulterior purpose, such as to make money, or to amend his life, or to benefit his fellows, he may be ever so much better than a scientific man, if you will -- to discuss that would be aside from the question -- but he is not a scientific man. For example, there are numbers of

> chemists who occupy themselves exclusively with the study of dyestuffs. They discover facts that are useful to scientific chemistry; but they do not rank as genuine scientific men. The genuine scientific chemist cares just as much to learn about erbium -- the extreme rarity of which renders it commercially unimportant -- as he does about iron. He is more eager to learn about erbium if the knowledge of it would do more to complete his conception of the Periodic Law, which expresses the mutual relations of the elements [CP 1.45, Peirce, 1994].

Charles Peirce found convenient to rename his philosophical standpoint from pragmatism into pragmaticism, his argument being:

> So then, the writer, finding his bantling "pragmatism" so promoted, feels that it is time to kiss his child good-by and relinquish it to its higher destiny; while to serve the precise purpose of expressing the original definition, he begs to announce the birth of the word "pragmaticism," which is ugly enough to be safe from kidnappers [CP 5.414, Peirce, 1994].

We will try to keep our view within Peirce's original view, knowing that every reader is a potential kidnapper of the term (the present authors included).

Peirce introduced the fundamental concept of **Reality** as follows:

> Such is the method of science. Its fundamental hypothesis, restated in more familiar language, is this: There are Real things, whose characters are entirely independent of our opinions about them; those Reals affect our senses according to regular laws, and, though our sensations are as different as are our relations to the objects, yet, by taking advantage of the laws of perception, we can ascertain by reasoning how things really and truly are; and any man, if he have sufficient experience and he reason enough about it, will be led to the one True conclusion. The new conception here involved is that of Reality [CP 5.384, Peirce, 1994].

Since reality is independent of the subject we can say that a fundamental requirement of the real is to be objective or at least intersubjective. While the observations are prone of circumstances such as where and when, as well as the observer, the *facts* have been usually deprived of such elements. The *ideated facts* are the reality in Peirce, and these facts are the point upon which two observers can agree [CP 6.522, Peirce, 1994]. We also learn about hypothesis in Peirce, although we prefer to use the name **conjectures**:

> A hypothesis is something which looks as if it might be true and were true, and which is capable of verification or refutation by comparison with facts. The best hypothesis, in the sense of the one most recommending itself to the inquirer, is the one which can be the most readily refuted if it is false. [CP 1.120, Peirce, 1994]

For Peirce, predictions are predictions of facts, since events/observations are haphazard [CP 2.752, 6.527, Peirce, 1994]. We will take a compatible but alternative view: we shall call **prediction** an expected observation based upon the known facts, the hypothesis h (conjectures), and logical/deductive elaborations. To produce a prediction we have to elaborate our known facts and, before verification, we have to provide the **particularities**[33] (unpredicted elements) that move us back to SW from IW. In mathematics, the map Π would be called a projection, and the predictive mapping, Γ, is named a "lift"[34], while we will use the symbol ϕ to denote the rational elaboration of ideas. When dealing with spontaneous observations, the need for Γ may appear unjustified, but if we are to conceive experiments to test a theory, Γ is a most relevant object that tells us what to expect, and it is this expectation what is really confronted against experimental results.[35] Figure 2.3.1 shows this outlined schema of science.

There are three conditions to be satisfied for the schema $\{\Pi, \phi, \Gamma\}$ to be consistent. If we ideate a set of particular observations –name the particularism by α– to construct a theory, $\Pi(\{Obs\}_\alpha)$, when we interpret the ideas in the theory using the same particularism, $\Gamma_\alpha\left(\Pi(\{Obs\}_\alpha)\right)$, we must recover the observed,

$$\Gamma_\alpha\left(\Pi(\{Obs\}_\alpha)\right) = \{Obs\}_\alpha \tag{2.3.1}$$

Thus, $\Gamma_\alpha \circ \Pi$ acts as the identity within particularism α. Correspondingly, if we produce the theory out of an interpreted set of concepts and relations, $\Pi\left(\Gamma_\alpha(\tau)\right)$, we should get the original set of concepts and relations τ constituting the theory

$$\Pi\left(\Gamma_\alpha(\tau)\right) = \tau \tag{2.3.2}$$

The above conditions should hold for any set of particular observations. Furthermore, if our theory shall not be considered refuted, we should have that

$$\Gamma_\beta\left(\phi(\Pi(\{Obs\}_\alpha)\right) \equiv \Gamma_\beta \circ \phi \circ \Pi\left(\{Obs\}_\alpha\right) = \{Obs\}_\beta \tag{2.3.3}$$

---

[33] We use particularities in the sense given by Peirce in the following expression:

> But observed facts relate exclusively to the particular circumstances that happened to exist when they were observed. They do not relate to any future occasions upon which we may be in doubt how we ought to act. They, therefore, do not, in themselves, contain any practical knowledge. [CP 6.523, Peirce, 1994]

[34] The word "lift" takes in mathematics a related but different meaning in the context of fibre bundles [Rotman, 1988]. Here Π is a projection that produces an abstract idea from observations, an idea that fits them all, and can rightly be called "the essence" of the phenomena. Thus, in $\Pi : SW \mapsto IW$ we may regard SW as a sort of product space and the projected portion is the set of circumstances or arbitrariness, $\{\alpha\}$. The image IW consists of the building blocks of the abduction process. Given the arbitrariness $\{\alpha\}$ and IW, Γ produces a reconstruction of SW. In mathematics, the lift would connect SW with $(IW, \{\alpha\})$. We use the word "lift" in a more colloquial form as the collection of inverses of Π restricted to the observations with arbitrariness α.

[35] Hertz [p. 20, 1893] makes this matter clear:

> Notwithstanding the greatest admiration for Maxwell's mathematical conceptions, I have not always felt quite certain myself of having grasped the physical significance of his statements. Hence it was not possible for me to be guided in my experiments directly by Maxwell's book.

When testing theories, experimental scientists need Γ.

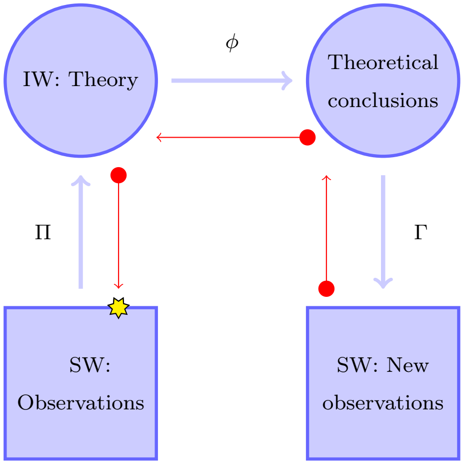

FIGURE 2.3.1. The proposed schema of mature Science. Π is a projection that produces the real out of the observed, ϕ stands for a theoretical elaboration (which eventually can be none, in such a case ϕ is the identity Id) and Γ is the interpretation that produces an expected observation.

i.e., that the theoretical conclusions elaborated by a set of observations with particularities α can be lifted to a corresponding set of observations $\{Obs\}_\beta$ with particularity β. For the special case in which ϕ presents no elaboration (i.e., $\phi = Id$), eq. (2.3.3) represents the standard requirement of reproducibility of experiments. The set of functions $\{\phi\}$ are automorphisms of IW while the transformations $T_{\beta\alpha}$ among particular representations,

$$T_{\beta\alpha} = \Gamma_\beta \circ \Pi\left(\{\cdot\}_\alpha\right)$$

are automorphisms of SW.

It is important at this point to realise that:

- The triple $\{\Pi, \phi, \Gamma\}$ can be designated with various (unfortunately ambiguous) names such as: schema and **theory**.
- The triple $\{\Pi, \phi, \Gamma\}$ depends on the conjectures h. We are avoiding to overload the notation and omit the symbol. In no case we are going to mix symbols belonging to different theories (since every triple $\{\Pi, \phi, \Gamma\}^h$can be said to be the theory based upon the conjecture h).
- In the case of incorporating conjectures we will request for them to have a set of testable consequences larger than those that motivated their inclusion. If possible, they should be associated with a cognitive surpass [Piaget and García, 1989] (see also [Solari and Natiello, 2018]). In

terms of [Kuipers, 1999] the aims are: theory revision (aiming to empirical progress), truth approximation and generalisation/abstraction (the latter not in Kuipers). Notice that higher-order theories are confronted with observations every time a particular lower order theory (associated to them in the dialectic universal-particular) is challenged. For example: we are testing logic almost all the time.

- It must also be kept in mind that $\{\Pi, \phi, \Gamma\}$ depends not only on the observations but it depends as well on the question being asked. An example will help to grasp the idea. Assume we have harvested m Red Delicious apples, of large size but wormy, and n Granny Smith apples of small size and healthy. If we ask how many apples there are, we will project out the variety, size and sanitary status, keep only the quantity, use addition of integer numbers and write our answer as: there are $m + n$ apples, some Granny Smith, small and healthy and some Red delicious, large but wormy. However, not being experts in apples, we are inclined to believe that if the question is about the price of the lot we will have to keep all the attributes to make a prediction and we will probably fail anyway since we are not including information about the market. Moreover, to propose an expression to evaluate prices with the information at hand we will have to advance a formula, i.e., introduce a conjecture. In this example $\{Obs\}_\alpha$ are the apples of the Sensorial World, α denotes the characteristics of size, health and sort, which are projected out by Π, leaving us with an integer number representing the apples of the Ideal World of the theory. ϕ is integer addition and Γ restores the characteristics projected out by Π.
- The elements of the triple are not independent and the theory is put into tension pulling from the "extremes". For a theory is constructed to answer some questions (the task of Γ) and data is procured to be plugged into a logical elaboration (ϕ).[36]
- Probabilistic theories are not different in our respect from other theories. The idealisation of random events institutes that they have to be considered only collectively (as idealised by Π), individual events are instances of evaluation of random variables (in the Γ version). We can only consider about them the set of allowed values and their probability distribution (in terms of frequencies). The logical and mathematical operations ϕ consent to elaborate answers from the premises in terms of probabilities. For example, quantum mechanics allows to predict the outcomes of experiments such as Stern-Gerlach's that relates detector counts with different dispositions in the setup. Population dynamics represents the probable state of the population in terms of integer numbers

[36] We often say "data is born contaminated with theory". Some things that are considered of no relevance are not measured (they are infinitely many, which is an excellent argument for not measuring them). At later times they may turn the experiment useless as their relevance becomes apparent.

and conciliates it with continuous time evolution resting on theorising about the evolution of probabilities of the population numbers.

- Peirce used abduction and retroduction indistinctly. Chiasson [2005] in his exegesis of Peirce argues that they should be distinguished as suggested by their etymological content, while "duction" means "lead" in latin "ab" stands for "away from" while "retro" means "going backward". Thus, moving away from exegesis, the backward flow of error merits the name of retroduction. Some parts of it are abductive as they open up the reasoning for different possibilities like in the implication $\{A, B\} \Rightarrow C$: when C evaluates to False, we have to admit that A or B are False, and we will have to confront at least three new hypothetical scenes. This means that the backward flow, the retroduction, calls for the formulation of new hypotheses as well as the collection of new kinds of data to discriminate among them, finally resulting in a more educated theory $\{\Pi, \phi, \Gamma\}^*$. This back and forth of construction, the iteration of successive reconstructions, has been presented in [García, 2011]. The iteration stops when all predictions evaluate to True. We can then rest, believing in the theory. The iteration recommences when an anomaly is discovered and doubt urges us again. "Learning from errors" means overcoming failure. Hence, failure must be thanked for giving us an opportunity of learning. In contrast, success –stopping the iteration of doubts and beliefs– puts it on a pause; we stop learning when successful.
- The triple $\{\Pi, \phi, \Gamma\}$ is a logical structure that allows to learn from mistakes. We discuss a historical example in Section 3.7
- Different theories may be compatible with the same observations. However, we are not comparing theories (or mixing them in any other form) but studying the process in which a given theory (under development) connects the Sensorial World with the Ideal World back and forth. Statements (2.3.1) and (2.3.2) indicate the need to be consistent in terms of the production of ideas and their interpretation. Idealisation and interpretation are not independent of each other .

The structure arising from the triple $\{\Pi, \phi, \Gamma\}$ corresponds to the idea of abduction-based reasoning, discussed in [CP 6.522–6.525, Peirce, 1994]. Let us consider it in more detail. The projection Π suggests the nature of the relevant observations. It is supported by epistemological frames and other beliefs –and probably by some conjectures. It transforms observations into facts in the language of Piaget and García [1989]. ϕ elaborates the ideation following the rules of mathematical logic and established theories and Γ confronts the elaborated ideas against the sensorial world. In this sense, Γ can be called *interpretation*, connecting the ideas with the Sensorial World. If the confrontation is successful, we preliminarily accept the associated conjectures as satisfactory explanations. If, on the contrary, the theory is refuted, we abandon it and proceed to generate a new, improved, conjecture in the light of the refutation, which in turn means a new theory $\{\Pi, \phi, \Gamma\}^*$. This last process is illustrated by the backwards path in Figure 2.3.1 indicating that disagreements between predictions and facts trigger improvements in Π. Also, if the theory does not contribute to organise and structure our views on the Sensorial World, it is rejected (it usually entails only to reject the conjectures). Conjectures that are impossible to test must be rejected as well [CP 1.71-1.74, Peirce, 1994]. Contemporary readers may recognise some of these ideas in Popper. Apart from precedence, Peirce goes further in his insight on Π.

The process of construction of a theory has been frequently misunderstood. At the second third of the XIXth Century a public debate on "Scientific knowledge" was held between John Stuart Mill and William Whewell [Strong, 1955]. A central part of the controversy revolved around the work of Johannes Kepler, one of the few philosophers that made his quest for knowledge transparent showing the intermediate results that lead to the final knowledge [Boido, 1998]. According to Peirce [CP 1.71-1.74, Peirce, 1994] Mill presents the view of a scientifically informed person, a vision of science from the outside, while Whewell presents the view from the inside. As Streintz did with Mach, Peirce gets to the point to sustain that Mill did not actually read Kepler's work "Mill certainly never read the *De Motu [Motibus] Stellae Martis*, which is not easy reading. The reason it is not easy is that it calls for the most vigorous exercise of all the powers of reasoning from beginning to end." We can see once again conflicting readings of the writings of the philosophers. Where Mill saw mere "facts" Whewell understood how reason was constructing the facts from the observations (data in this case). Peirce writes about Kepler (CP 1.74):

> Thus, never modifying his theory capriciously, but always with a sound and rational motive for just the modification he makes, it follows that when he finally reaches a modification - of most striking simplicity and rationality - which exactly satisfies the observations, it stands upon a totally different logical footing from what it would if it had been struck out at random, or the reader knows not how, and had been found to satisfy the observation. Kepler shows his keen logical sense in detailing the whole process by which he finally arrived at the true orbit. This is the greatest piece of Retroductive reasoning ever performed.

Since there are different uses of the word "abduction" corresponding to different readings of Peirce, we will abound with another example. Consider an individual experiencing fever, headache and muscular ache. COVID-19 is the prevalent disease in their town these days. They consults a medical doctor who orders some tests: one that would detect signs of COVID-19, another detecting signs of flu and a third one detecting bacterial infections. Each of the hypotheses h proposes some projection, Π_h, of the illness into the associated biochemistry. The tests are ordered because there is some logic and prior knowledge, ϕ, that allows to elaborate each hypothesis into distinctive/discriminant biochemical outcomes for the test. The outcomes of the tests are considered confirmation (True) or rejection (False) of the various hypotheses. It is at this point, when the test is completed, that the process of abduction ends. It is not just Π. Nor it can be reduced to the guessing work performed (vulgar inference) even when the expectations on the hypothesis could be weighted by the current prevalence of illnesses. Probabilities might have biased our guessing work, but if we get a negative result for all the tests, we have learned something and our guessing work will be subsequently modified to produce new hypothesis compatible with the new information. In this respect, a proposition such as

> Ignorance-preservation in the context of ignorance problems: Whereas deduction is truth-preserving and induction is probability-enhancing, abduction is ignorance-preserving [Gabbay and Woods, 2006]

indicates that either our approach of constructing scientific theories cannot be classified as an "ignorance problem" or that we associate a different meaning to "abduction". It can easily be verified that at least the meaning of abduction is different. Let us emphasise the recursive, consistent and constant use of the method of analysis.

Considered as machines, humans are logic machines that control their own logic, which brings to mind the conclusion brought by Burks [1946] regarding Peirce's thoughts on logical machines:

> A computing machine starting from correct premises can arrive at correct conclusions. In what respects, then, does such a machine fail to infer? In the first place, the procedure of a machine does not have the element of conscious approval and control; it cannot certify the validity of its own inferences. Secondly, a computing machine lacks originality.

Neither Π nor Γ are deductive or inductive. Concerning Γ, it maps $IW \mapsto SW$, while induction goes in the same direction as Π (namely $SW \mapsto IW$) and deduction connects $IW \mapsto IW$. Hence, Γ is not induction since it has a different domain and it is not deduction since it has a different image. The only truly deductive part of the process lies in ϕ. Science cannot be reduced to logic only. There are three types of reasoning [McAuliffe, 2015] in Peirce:

> These three kinds of reasoning are Abduction, Induction, and Deduction. Deduction is the only necessary reasoning. It is the reasoning of mathematics. It starts from a hypothesis, the truth or falsity of which has nothing to do with the reasoning; and of course its conclusions are equally ideal [...] Induction is the experimental testing of a theory. The justification of it is that, although the conclusion at any stage of the investigation may be more or less erroneous, yet the further application of the same method must correct the error. The only thing that induction accomplishes is to determine the value of a quantity. It sets out with a theory and it measures the degree of concordance of that theory with fact. It never can originate any idea whatever. No more can deduction. All the ideas of science come to it by the way of Abduction. Abduction consists in studying facts and devising a theory to explain

> them. Its only justification is that if we are ever to understand things at all, it must be in that way. [CP 5.145, Peirce, 1994]

The role of interpretation, as presented here, resembles the one attributed to "induction" in Peirce. The reason to change names is that most of us associate "induction" with a process that goes from observations to ideas, while interpretation goes in a somewhat opposite direction: From general ideas to particulars, as already explained.

Afterthought on probabilities. While the present work is mostly concerned with the philosophical idea of science, this is, the science that belongs in IW, there are other forms of science pertaining to SW. The social use of science quite often rests on theories being currently developed, this is: in construction. In such cases, Π and Γ_α are frequently based upon observations that do not fully explore the universe of particular situations. Thus, these theories have hidden variables outside the control of consciousness, or oversimplified relations between variables (laws). In such situations the predictions some times are correct and some times are incorrect. From the point of view of the construction of the theory the situation indicates incomplete understanding, and the theory is refuted; meaning that more research is needed. From a practical point of view, even an imperfect theory may have social value. In such cases, it will present its predictions in terms of probabilities.

Afterthought on equivalence/inequivalence of theories. We say that two theories $\{\Pi, \phi, \Gamma\}$ and $\{\Pi', \phi', \Gamma'\}$ are equivalent if there exists Q invertible such that $Q \circ \Pi = \Pi'$ and $Q \circ \phi \circ \Pi = \phi' \circ \Pi'$, i.e., $\phi' \circ Q = Q \circ \phi$. This means that every idealised expression in one theory has a corresponding counterpart in the other theory. Further, we say that two theories explain all the observable (actually observed or potentially observable) if for every α, $\Gamma'_\alpha \circ (\phi' \circ \Pi') = \Gamma_\alpha \circ (\phi \circ \Pi)$, i.e., all predictions in one theory correspond to a prediction in the other theory with the same outcome and tracing back to the same object in SW. Hence, it follows that

Theorem 2.1. *Two theories explaining all the observable (actually observed or potentially observable) are equivalent.*

Proof. Let $Q = \Pi' \circ \Gamma_\alpha$. By eq. (2.3.1), $Q \circ \Pi = (\Pi' \circ \Gamma_\alpha) \circ \Pi = \Pi'$. Further, by eq. (2.3.2) $Q \circ \phi \circ \Pi = \Pi' \circ \Gamma_\alpha \circ (\phi \circ \Pi) = \Pi' \circ \Gamma'_\alpha \circ (\phi' \circ \Pi') = \phi' \circ \Pi'$. That Q is invertible follows from noting that $Q^{-1} = \Pi \circ \Gamma'_\alpha$. □

Corollary 2.1. *Two theories are different (not equivalent) if there exists at least one experiment with different proposed/theorised outcome (be it that we expect different results or that the experiment can be explained in one theory but cannot be conceived in the other).*

This result relates to the idea of "empirical equivalence" [Duhem, 1991] although here it is used in the opposite direction, namely illustrating that distinction among theories that exactly coincide in all observable tests is unnecessary and irrelevant, as far as appraisal of theories rests on the contrasting/refuting process. Equivalent theories differ in arbitrary choices not relevant for the contrasting/refuting process (by Corollary 2.1) and that the *real* (what the rational support of the theory deals with) is what is left after eliminating arbitrariness.

In the present context, as a consequence of Corollary 2.1, discriminating one equivalent theory from another on the basis of "Inference to the best [potential] explanation" [Boyd, 1983, Lipton, 2004] appears as subjective. Rather, preferring one equivalent theory to another reveals the kind of argumentation that scientists may consider more appealing. Apparently, this is what happened in the late XIXth. Century when physicists massively linked words such as "understanding" and "explaining" to one particular form of argumentation (see Section 3.2) rejecting other forms that were completely equivalent in terms of the experimental knowledge of their time.

Appendix

2.A. Scholium. A "pragmaticist" criticism of Special Relativity

We want to illustrate in this section how the epistemological frame[37] [Piaget and García, 1989] changes our appraisal of theories. We address the problem in practical terms considering special relativity, SR, one of the theories that prompted the need for a new epistemology as presented by Popper:

> [Referring to other epistemological approaches] They will hardly be ready to grant this dignity to modern theoretical physics in which I and others see the most complete realization to date of what I call 'empirical science'. [...] Thus I freely admit that in arriving at my proposals I have been guided, in the last analysis, by value judgments and predilections. [p. 15, Popper, 1959]

Poincaré [1905] introduced the Principle of relativity writing:

> But it is not sufficient that the Euclidean (or non-Euclidean) geometry can ever be directly contradicted by experiment. Nor could it happen that it can only agree with experiment by a violation of the principle of sufficient reason, and of that of the relativity of space. Let me explain myself. Consider any material system whatever. We have to consider on the one hand the "state" of the various bodies of this system—for example, their temperature, their electric potential, etc.; and on the other hand their position in space. And among the data which enable us to define this position we distinguish the mutual distances of these bodies that define their relative positions, and the conditions which define the absolute position of the system and its absolute orientation in space. The law of the phenomena which will be produced in this system

[37] Generally speaking, an epistemological frame provides an a priori form of organising concepts, formulating questions, producing and presenting answers. As such, it can discard questions as unsuitable or lacking interest and even produce "blindness" as in the example presented in [García, 1981]. Moreover, the same question/answer might have completely different readings depending on the epistemological frame.

> will depend on the state of these bodies, and on their mutual distances; but because of the relativity and the inertia of space, they will not depend on the absolute position and orientation of the system. In other words, *the state of the bodies and their mutual distances at any moment will solely depend on the state of the same bodies and on their mutual distances at the initial moment, but will in no way depend on the absolute initial position of the system and of its absolute initial orientation.* This is what we shall call, for the sake of abbreviation, the law of relativity. [Emphasis added][38]

In [Einstein, 1905a] the principle is presented as a conjecture, raised to the level of postulate:

> The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.

The relation between Poincaré's version and Einstein's can readily be seen from the quoted text. Einstein's presentation of the law is operational. He dispenses of the intuitions and foundations presented by Poincaré. In particular, for Poincaré relative distances appear as real, a fundamental intuition, while in Einstein they will become apparent.

The present pragmaticist view indicates that the Principle of Relativity in classical mechanics is not a new or independent principle, but rather the consequence of requiring a rational foundation for our understanding and therefore eliminating arbitrariness. It is supported in the relational view of mechanics that goes back to Leibniz [Solari and Natiello, 2018] and we have called it the No Arbitrariness Principle (NAP). In some sense, it integrates Newton's mechanics with Leibniz' objections, thus extending beyond both. This form of surpass imposes that the mappings connecting presentations of dynamical processes under different arbitrary conditions constitute a *group* (a mathematical structure of associative binary operations having inverse and identity operation). In classical mechanics one of the groups relating arbitrary (subjective) choices is the group of Galilean coordinate transformations, eliminating the arbitrariness in the relative motion between reference systems. Relativity proposes to replace Galileo's transformations by Lorentz transformations (LT). Under the pragmaticist schema it is then necessary that the LT's constitute a group (which they don't) and that they eliminate the corresponding arbitrariness in the relative motion between reference systems.

[38] The paragraph illustrates as well the difficulties with the concept of "inertial systems". Poincaré rests upon first conceiving absolute space and next suppressing its influence. Instead of saying that there are no such things as absolute position and absolute orientation, he writes "...because of the relativity and the inertia of space, they will not depend on the absolute position and orientation...". Absolute space appears as an auxiliary metaphysical concept for thoughts, a first reference of space, that later will have to be suppressed. The discussion of the concept of inertial systems is part of the transition from classical to current physics [Solari and Natiello, 2021].

Nothing is gained by enlarging these transformations with the full Poincaré-Lorentz group (PL), the arbitrariness cannot be fully eliminated.

The questions are: which is the residual arbitrariness that remains? How was it introduced in special relativity?

These questions examine the axioms and the inference (abduction) leading to the axioms, i.e., they concern the projection Π, before refutation or verification can enter the discussion. Therefore its criticism lies outside Popper's epistemic approach and outside the "orthodox" approach (as described by Feigl) as well.

2.A.1. On the projection Π involved in Special Relativity. In the first paragraph of Einstein's fundamental paper [Einstein, 1905a] it is stated that physical phenomena depend on relative motion of the interacting parts, suggesting that relative velocity is a well-defined concept. This is a fundamental assumption of the theory that echoes the known properties of relative velocity in classical mechanics. This idea is completed in Part I, §3 stating that if a system k moves with velocity v with respect to a system K, and a system K' moves with velocity $-v$ with respect to k, then K and K' are at rest relative to each other. Finally, the change of coordinates between k and K is shown to be an invertible Lorentz transformation (LT), let's name it $L_v = L_{-v}^{-1}$.

In the terms of this work, relative velocity should be the outcome of Π or a mathematical result constructed combining outcomes of Π, it should be in the image of the observable. Recall also that to be properly defined, relative velocity between e.g., two bodies A and B must be a quantity depending only on A and B without intervention of other observers, frames or references, a demand expressed by Definition 1.1, points 1 and 4.

While in [Einstein, 1905a] one of the systems is called "stationary" (meaning that v is the velocity of the "other" system as described by the stationary one), later [p. 514, Einstein, 1907] habilitates the LT's as a change of (space-time) coordinates for systems in relative motion at constant velocity. Hence, according with the claim, for a reference frame S_0 plus a set of reference frames $S_i,\ i \geq 1$ each one moving with velocity u_i with respect to S_0 and all sharing the same origin of space-time coordinates, we have that $(x_i, y_i, z_i, t_i) = L_{u_i}(x_0, y_0, z_0, t_0)$. Since L_{u_i} is invertible, the transformation from reference frame S_i to reference frame S_j is $(x_j, y_j, z_j, t_j) = \left(L_{u_j} \circ L_{-u_i}\right)(x_i, y_i, z_i, t_i)$. According to the conjecture that the reference systems are of the same kind, there should be a relative velocity u_{ji} so that its associated LT satisfies $\left(L_{u_j} \circ L_{-u_i}\right) = L_{u_{ji}}$. This is to say that the generalisation from [Einstein, 1905a] to [Einstein, 1907] has the hidden hypothesis that the LT's form a group, requiring that the successive applications of two such transformations is a LT as well. It is known that for any pair of velocities such that $u_i \times u_j \neq 0$ this is not the case, see for example [Silberstein, 1914, Gilmore, 1974]. Hence, the hidden assumption must be rejected.

As a consequence, it is possible to put to experimental test Einstein's 1905 theory in this regard, as for example is done in Doppler experiments [Dingle, 1960b, Kaivola et al., 1985, Mandelberg and Witten, 1962] with the theory surviving the test. In such case, the relative velocity must be computed using the standard methods of classical mechanics (since one of the systems is "stationary").

On the other hand, to put the 1907 rewriting of the theory to this test it is required to show how the map Π is used in order to compute the relative velocity between two bodies, say A and B, knowing their relative velocity as measured with respect to a third system S. Given the velocities v_{AS} and v_{BS} of A and B relative to S as initial data, there is no way in SR to establish v_{AB} by coordinate transformations as an outcome that does not involve S. The only candidate in the framework of Lorentz transformations (or even in the broader framework of the whole Poincaré-Lorentz group) would be $v_{AB} = v_{AS} \oplus (-v_{BS})$ arising from the law of addition of velocities [p. 168, Silberstein, 1914], but (a) this quantity is intrinsically dependent on S, (b) it fails to satisfy the fundamental demand of reversibility, since $v_{AB} \neq -v_{BA}$ and (c) it is not unique. In short, there is no prescription on Π that can relate the observable to the idealised under the indicated conjectures and the theory is not testable and should be rejected in terms of the epistemology supporting the phenomenological map.

2.B. Scholium. The phenomenological map

The phenomenological map connects and at the same time keeps as distinguished the world of ideas (IW) and the world of observations (SW). For critical philosophy, cognition requires both, as in Kant's dictum:

> Understanding cannot intuit, and the sensuous faculty cannot think. In no other way than from the united operation of both, can knowledge arise.

The phenomenological map that represents the dialogue between the sensuous faculty and our thoughts separates them and prevents hypostatisation, the reification of ideas. Not only Kant spoke against hypostatisation in his discussion of metaphysics. Faraday, an Experimental Philosopher, wrote:

> But it is always safe and philosophic to distinguish, as much as is in our power, fact from theory; the experience of past ages is sufficient to show us the wisdom of such a course; and considering the constant tendency of the mind to rest on an assumption, and, when it answers every present purpose, to forget that it is an assumption, we ought to remember that it, in such cases, becomes a prejudice, and inevitably interferes, more or less, with a clear-sighted judgment. [p. 285, Faraday, 1844]

We finish this Chapter with Kant's words (we invite the reader to substitute "pure reason" by "the sciences" in the next quotation)

> Reason must be subject, in all its operations, to criticism, which must always be permitted to exercise its functions without restraint; otherwise its interests are imperilled and its influence obnoxious to suspicion. There is nothing, however useful, however sacred it may be, that can claim exemption from

the searching examination of this supreme tribunal, which has no respect of persons. The very existence of reason depends upon this freedom; for the voice of reason is not that of a dictatorial and despotic power, it is rather like the vote of the citizens of a free state, every member of which must have the privilege of giving free expression to his doubts, and possess even the right of veto.

But while reason can never decline to submit itself to the tribunal of criticism, it has not always cause to dread the judgment of this court. Pure reason, however, when engaged in the sphere of dogmatism, is not so thoroughly conscious of a strict observance of its highest laws, as to appear before a higher judicial reason with perfect confidence. On the contrary, it must renounce its magnificent dogmatical pretensions in philosophy. [p. 475, Kant, 1787]

CHAPTER 3

Change in the epistemic praxis. The ether (hypotheses are welcome)

3.1. Introduction

While the previous Chapter links the Enlightenment tradition to Whewell's and Peirce's ideas and to our phenomenological map describing a rational approach to Science, history followed a different path and the Enlightenment heritage was abandoned in favour of an utilitarian approach somewhere around 1870, following the second industrial revolution. In this Chapter we will describe this transition, showing how social forces and interests other than the love of Nature were in action.

An outer view of science has existed probably since science exists. The outer view corresponds to those that follow the advances of science without being themselves philosophers that contribute to science. By the times of Galileo, Francis Bacon wrote an influential treatise, the *Novum organum*, or "True Suggestions for the Interpretation of Nature". Bacon, a philosopher and politician, mixed his political thinking with his outer knowledge of Science, as for example in:

> It will, perhaps, be as well to distinguish three species and degrees of ambition. First, that of men who are anxious to enlarge their own power in their country, which is a vulgar and degenerate kind; next, that of men who strive to enlarge the power and empire of their country over mankind, which is more dignified but not less covetous; but if one were to endeavor to renew and enlarge the power and empire of mankind in general over the universe, such ambition (if it may be so termed) is both more sound and more noble than the other two. Now the empire of man over things is founded on the arts and sciences alone, for nature is only to be commanded by obeying her. [Bacon, 1902]

Bacon's view clearly considers Science as a tool for conquering Nature. It is not love to Nature what moves him but his expectations about mankind's interest: knowing nature to command it. For about two hundred years Science was advanced by love rather than by interest, yet, the idea of dominating Nature and profiting from such domination was entertained by the "powers that be" (the ruling class in Marxist jargon).

During the XIXth Century in Europe the development of the second industrial revolution benefited from science. Carnot, Kelvin and Clausius (among others) contributed to the understanding of thermodynamics and the improvement of heat engines (the original aim of Carnot's research [Kondepudi and Prigogine, 2014]). The "Telegraph Construction Firm of Siemens & Halske" was founded in Berlin 1847[39] and it soon expanded its business into High Voltage Electrical engineering. According to Ortega y Gasset:

> This 19th Century civilisation, I said, can be summed up in two major dimensions: liberal democracy and technology. Let us now take only the latter. Contemporary technology was born out of the copulation between capitalism and experimental science. Not all technology is scientific. The maker of flint axes in the Chelonian period lacked science, and yet he created a technique. [Chapter XII, Ortega y Gasset, 1930]

The passion for learning that moved Peirce's scientist had to accommodate itself with the interest in knowledge, a product that came coded in mathematical formulae for physics. But, if what really matters are the formulae, the way we arrive to them is just anecdote. By 1874 Boltzmann writes in "On the development of the methods of theoretical physics in recent times" (1899):

> Hertz was not looking for a satisfactory mechanical explanation of these fundamental equations, at least he did not find one; but he even spurned the Euclidean mode of derivation. He rightly points out that what convinces us of the correctness of all these equations is not, in mechanics, the few experiments from which its fundamental equations are usually derived, nor, in electrodynamics, the five or six basic experiments of Ampère, but rather their subsequent agreement with almost all hitherto known facts. He therefore passes a judgment of Solomon that since we have these equations we had best write them down without derivation, compare them with phenomena and regard constant agreement between the two as the best proof that the equations are correct.
>
> The view whose most extreme form has here been stated, was very variously received. Whereas some were almost inclined to regard it as a bad joke, others felt that physics must henceforth pursue the sole aim of writing down for each series of phenomena, without any hypothesis, model or mechanical explanation, equations from which the course of the phenomena can be quantitatively determined; so that **the sole task of physics consisted in using trial and error to find the simplest equations that satisfied certain required formal conditions of isotropy and so on, and then to compare them with experience**. This is the most extreme

[39] https://www.britannica.com/money/Siemens-AG accessed January 14th 2025.

> form of phenomenology, which I should like to call mathematical, whereas general phenomenology seeks to describe every group of facts by enumeration and by an account of the natural history of all phenomena that belong to that area, without restriction as to means employed except that it renounces any uniform conception of nature, any mechanical explanation or other rational foundation. This latter view is characterized by Mach's dictum that electricity is nothing but the sum of all experience that we have had in this field and still hope to have. Both views set themselves the task of representing phenomena without going beyond experience.[p. 94–95, Boltzmann, 1974] (Emphasis added).

In this extreme view it is only the product what matters and as any other product its quality is asserted by its accumulated success in performing as expected. The word success, a word that "the men who carry on the business of the world" truly understand and appreciate, enters the scene of science. As any other business, Science is a matter of economy. Recall Mach's claim on thought experiments (see Footnote 26 in Chapter 1).

This approach to knowledge and science has been called *instrumentalism.* It is the characteristic approach for the period starting around 1870 and it has been the driving force in technological developments. We will return to its relation with science in Section 3.4.1 and Chapter 5.

Science cannot escape the ethos of its time. Perhaps the famous words of the Communist manifest resume it best:

> The bourgeoisie, historically, has played a most revolutionary part. The bourgeoisie, wherever it has got the upper hand, has put an end to all feudal, patriarchal, idyllic relations. It has pitilessly torn asunder the motley feudal ties that bound man to his "natural superiors", and has left remaining no other nexus between man and man than naked self-interest, than callous "cash payment". It has drowned the most heavenly ecstasies of religious fervour, of chivalrous enthusiasm, of philistine sentimentalism, in the icy water of egotistical calculation. It has resolved personal worth into exchange value, and in place of the numberless indefeasible chartered freedoms, has set up that single, unconscionable freedom – Free Trade. In one word, for exploitation, veiled by religious and political illusions, it has substituted naked, shameless, direct, brutal exploitation. The bourgeoisie has stripped of its halo every occupation hitherto honoured and looked up to with reverent awe. **It has converted** the physician, the lawyer, the priest, the poet, **the man of science, into its paid wage labourers.** [Marx and Engels, 2010] (Emphasis added)

3.2. The *second physicist* and analogical thinking

During the second half of the XIXth Century the number of physicists in the German universities doubled [Jungnickel and McCormmach, 2017]. A second professor of physics was appointed but, in general, no new laboratories were created. The second physicist, often known as mathematical-physicist, became finally known as theoretical physicist and was in charge of the mathematisation, previously the task of mathematicians and philosophers. Helmholtz explains:

> The change, in the Universities, to their present constitution, was caused mainly by the fact that the State granted to them material help, but required, on the other hand, the right of co-operating in their management.[p. 242, Helmholtz, 1908][...]
>
> The development of the German Universities differs characteristically from these two extremes. Too poor in their own possessions not to be compelled, with increasing demands for the means of instruction, eagerly to accept the help of the State, and too weak to resist encroachments upon their ancient rights in times in which modern States attempt to consolidate themselves, the German Universities have had to submit themselves to the controlling influence of the State. Owing to this latter circumstance the decision in all important University matters has in principle been transferred to the State, and in times of religious or political excitement this supreme power has occasionally been unscrupulously exerted. But in most cases the States which were working out their own independence were favourably disposed towards the Universities; [p. 247-248, Helmholtz, 1908][...]

The task of mathematicians at producing physical theories had been transferred to a new specialist: the theoretical physicist [Jungnickel and McCormmach, 2017] and furthermore, philosophers no longer exercised critical thinking in matters of science, at least in Germany. From Kant, through Reinhold, Fichte until Hegel, the prevalent movement considered that philosophy was "'the guardian of the sciences,' their founder and systematizer" [p. 15, Beiser, 2014]. But the movements that emerged after Hegel were mostly anti-Hegelian. Peirce comments (from Lessons on the History of Science, ca. 1896):

> German universities for a whole generation turned the cold shoulder to every man who did not extol their stale Hegelianism, until it became a stench in the nostrils of every man of common sense. Then the official fashion shifted, and a Hegelian is today treated in Germany with the same arrogant stupidity with which an anti-Hegelian formerly was.[CP 1.77, Peirce, 1994]

For the new German philosophy the task of philosophy with respect to the sciences ranged from studying "the logic of the science" as acted by the scientists, to the extreme of the sciences declaring philosophy dead, as in "...neo-Kantian Jürgen Bona Meyer: 'The daughters now demand independence from their common mother, and they do not suffer it gladly when they are supervised or corrected; they would prefer that their old and morose mother lay herself to rest in her grave'" (quotations from [pp. 17-18, Beiser, 2014]).

However, social changes were not the only influences for a change in epistemic praxis. The difficulties in getting a unified picture of electromagnetism were an important element as well. By 1856 a young Maxwell [Maxwell, 1856] exposed these problems in front of the Royal Society. Maxwell proposed in the conference a method for gaining insight into possible formulations based up on, what he named, "physical analogy". As explained by Maxwell, "physical analogy" consisted in establishing a correspondence between two (completely) different physical systems which however project into the same equations, as for example Professor William Thomson had shown between heat and attraction. Such method, if the correspondence is precise, makes the deductions of one theory conjugated to the deductions of the other. However, "Analogy is the inference that a not very large collection of objects which agree in various respects may very likely agree in another respect. For instance, the earth and Mars agree in so many respects that it seems not unlikely they may agree in being inhabited" [CP 1.69, Peirce, 1994]. Thus, analogy is good for suggesting ideas, but not enough to sustain a correspondence.

Scientists as Lord Kelvin and Maxwell supported their thoughts with analogies. Lord Kelvin writes

> I never satisfy myself until I can make a mechanical model of a thing. If I can make a mechanical model I can understand it. As long as I cannot make a mechanical model all the way through I cannot understand; and that is why I cannot get the electromagnetic theory. [p. 835, Thompson, 2011].

while Maxwell says:

> Now we are unable to conceive propagation in time, except either as the flight of a material substance through space, or as the propagation of a condition of motion or stress in a medium already existing in space.[...] If something is transmitted from one particle to another at a distance, what is its condition after it has left the one particle and before it has reached the other? [[866], Maxwell, 1873]

Indeed, light-travel is understood by analogies with bodies (we return to this idea in Section 3.6), still in our days, like a stone thrown by the source and captured by the detector. Also, electromagnetic waves were conceived mechanically and asked for a propagation medium that could sustain them after having abandoned the source and before reaching the detector.

A most decisive epistemological change was advanced by Hertz, a disciple of Helmholtz, who acted the idea that it is possible to separate the process of construction of a theory from the theory's mathematical content. Regarding Maxwell's electrodynamics, he states [p. 21, Hertz, 1893],

> "To the question: 'What is Maxwell's theory?'. I know of no shorter or more definite answer than the following:– Maxwell's theory is Maxwell's system of equations. Every theory which leads to the same system of equations, and therefore comprises the same possible phenomena, I would consider as being a form or special case of Maxwell's theory;[...]"

Hertz claimed that theories have to be provided with what he called *interpretation*, an element that runs alongside the mathematics and helps in constructing experiments related to theory (the action of Γ in Section 2.3) while at the same time theory can be detached from its construction. Several inequivalent interpretations can, in this form, be attached to a theory. Hertz' deep epistemological change has been highlighted by D'Agostino:

> ...by separating the mathematical structure of a theory from its modes of representation he [Hertz] has profoundly challenged the conception of a physical theory as an indivisible unity of the two – a conception accepted by Maxwell and other nineteenth century mathematical physicists." [D'Agostino, 1968, D'Agostino, 2004].

Einstein, in his attempt to solve the problems of electromagnetism associated to the ether, yet resting on Maxwell's alternative concerning "propagation in time", opted for "flight of a [immaterial] substance", which he named "photon". In this way he constructed a mechanical analogy and further, he introduced the principle of relativity as an Axiom [Einstein, 1905a], this is without justification other than an analogy of forms (or perhaps just an habit) founded on habits acquired with Newton's mechanics. As explained in Subsection 2.A.1, there is a hidden hypothesis that goes unperceived because of habituation and simple intuition, namely that between two reference systems a unique velocity can be defined which stands for the relative velocity among them. However, the new structure of space-time proposed by Einstein has no room for such a velocity. In addition, the idea that there is a set of reference systems where the laws of physics hold in equivalent form, requires (because of the notion of equivalence) that these systems form a class and then the transformations are required to form a group, which is not the case for Lorentz transformations.

Currently most of those claiming to adhere to a relational view actually adhere to a formal procedure that has emptied the word of meaning, since instead of seeking for reality they make room for a world of incommensurate subjective opinions (see [Margenau and Mould, 1957] for a distinction between an "older meaning" and a "modern form" of relativity). Even worse, the same can be said of "critic". How can science exercise criticism when it does not allow to search for its own fundamentals but considers only its consequences? In particular, the consistency of the triple $\{\Pi, \phi, \Gamma\}$ (See Chapter 2) is set aside, since both ends are debilitated or absent. Theories should be exposed to experimental analysis, but they may be rejected even earlier, if the phenomenological map is inadequate for the problem.

In today's formation of professional physicists analogies are central. In this sense, anthropologist Sharon Traweek in her observation of the high-energy physicists community says:

> Undergraduate physics students, to be successful, must display a high degree of intellectual skill, particularly in analogical thinking. The students learn from textbooks whose interpretation of physics is not to be challenged; in fact, it is not to be seen as interpretation. [p. 74, Traweek, 1992][...] Teachers show students how to recognize that a new problem is like this or that familiar problem; in this introduction to the repertoire of soluble problems to be memorized, the student is taught not induction or deduction but analogic thinking. (p. 77)

The observation gives a fair idea on how this form of thinking is trained and selected and shows how the epistemic frame is socially reproduced.

When abduction is conceived solely as the creative process of producing conjectures, analogy can be considered a form of abduction. However, when abduction is considered, as we do, the process of producing hypothesis, elaborating from them and confronting the results at empirical tests, concluding the abduction process by accepting non refuted conjectures, analogies are only a possible mechanism in the proposing part of the process.

3.2.1. Göttingen vs. Berlin. The transition to the utilitarian approach in Germany was centered around Berlin university, while Göttingen had been the guardian of the Enlightenment approach through Gauss and his disciples, namely Weber, Riemann, Neumann and Lorenz. They supported the epistemological approach prevailing since Galileo Galilei until Maxwell (died 1879), which we designate as "traditional".

The Berlin school of Physics, with Helmholtz, Clausius and Hertz among others, represented the "new" approach (to put a date, 1871 with Helmholtz becoming professor of physics). The transition time was then also the epoch in which the belief in a material ether was prevalent (we defer the discussion of the ether to Sections 3.6 and 3.7). This school introduced the *Bild* approach [Dieter, 1998, Heidelberger, 1998, Hoffmann, 1998, D'Agostino, 2004, Schiemann, 1998], a form of understanding supported in images or sensory signs. As a brief explanation:

> For Hertz, in contrast [with Helmholtz], representations of theories are signs of sensory impressions that are given to us. Only if we use theory to construct representations will it accomplish the most important task of natural knowledge, foresight of the future from experiences of the past. [Heidelberger, 1998]

Conceptually, since mental images are the outcome of sensorial perception, using images to organise understanding implies to allow sensory-based intuition to be used by analogy in other realms.

Basically, the Berlin legacy consisted in separating the equations from the construction of theories, leaving the latter outside scrutiny and to incorporate interpretations or "mental images" and analogies to connect the equations to experimental science. Thus, The value of a theory lies solely in its success as predictive tool.

3.3. The crisis

According to Poincaré, there was a crisis in mathematical physics by 1904 [original of September 1904, Poincaré, 1913a], and indeed there was one. Poincaré made an attempt to rescue some of the lessons of the old science (as he called it) in terms of principles that have been established by the old science and appeared to him as foundational. Among them the principle of relativity, the principle of minimal action, conservation of energy, Carnot's 2nd law of thermodynamics and a few others. But he realised that these principles themselves were in crisis as well. It is interesting from the point of view of the present work to quote some of his words regarding the crisis of the principle of relativity [original of September 1904, Poincaré, 1913a]

> Let us pass to the principle of relativity: this not only is confirmed by daily experience, not only is it a necessary consequence of the hypothesis of central forces, but it is irresistibly imposed upon our good sense, and yet it also is assailed. Consider two electrified bodies; though they seem to us at rest, they are both carried along by the motion of the earth; an electric charge in motion, Rowland has taught us, is equivalent to a current; these two charged bodies are, therefore, equivalent to two parallel currents of the same sense and these two currents should attract each other. In measuring this attraction, we shall measure the velocity of the earth; not its velocity in relation to the sun or the fixed stars, but its absolute velocity.
>
> I well know what will be said: It is not its absolute velocity that is measured, it is its velocity in relation to the ether. How unsatisfactory that is! Is it not evident that from the principle so understood we could no longer infer anything? It could no longer tell us anything just because it would no longer fear any contradiction. If we succeed in measuring anything, we shall always be free to say that this is not the absolute velocity, and if it is not the velocity in relation to the ether, it might always be the velocity in relation to some new unknown fluid with which we might fill space.

Here, Poincaré's criticism is aligned with Peirce's position, at least inasmuch he refuses to make non-refutable hypothesis.

The crisis identified by Poincaré proceeded with a new turn. By allowing free interpretations, the velocities involved in electromagnetism, that had been fully measurable relational velocities in Ampère, Faraday and Weber's experiments as well as in Maxwell's abduction, were reinterpreted in different ways. In their expressions of forces, and the derivation of them, Maxwell works with relative velocities between circuits [Maxwell, 1873]. Lorentz [1892] reinterpreted relative velocities as absolute velocities referring to the ether in its expression of the electromagnetic force. Later Einstein [1905a] kept Lorentz' expression of the force while eradicating the ether, which implies another reinterpretation of the involved velocities as velocities with respect to an inertial frame, and proposing to use Lorentz transformations in place of Galilean boosts to restore the relativity principle. In so doing, he was loyal to Hertz epistemological point of view, changing the interpretation of the glyphs of Maxwell's electrodynamics without changing the glyphs of the theory.

We have chosen Popper's "Logic of Scientific Research" to represent the alternative to the epistemology schematically presented in Chapter 2, as it evolved from the Berlin school. Popper's view of the philosophy of sciences is well aligned with the post-Hegelians:

> I suggest that it is the task of the logic of scientific discovery, or the logic of knowledge, to give a logical analysis of this procedure; that is, to analyse the method of the empirical sciences.[p. 4, Popper, 1959]

From pages 3 to 7 he address the problem of induction, concluding that induction is not the support of trust in science. He next writes against psychologism,

> The initial stage, the act of conceiving or inventing a theory, seems to me neither to call for logical analysis nor to be susceptible of it. The question how it happens that a new idea occurs to a man— whether it is a musical theme, a dramatic conflict, or a scientific theory—may be of great interest to empirical psychology; but it is irrelevant to the logical analysis of scientific knowledge. (p. 7) [...]

returning recurrently to his main thesis:

> According to the view that will be put forward here, the method of critically testing theories, and selecting them according to the results of tests, always proceeds on the following lines. From a new idea, put up tentatively, and **not yet justified in any way**—an anticipation, a hypothesis, a theoretical system, or **what you will**—conclusions are drawn by means of logical deduction. These conclusions are then compared with one another and with other relevant statements, so as to find what logical relations (such as equivalence, derivability, compatibility, or incompatibility) exist between them. (p. 9, emphasis added)

We have highlighted two expressions that give the clear impression that for Popper theories come from nowhere, leaving the process of production as not belonging to science. In short, for Popper the phenomenological moment appears as nonscientific.

By rejecting induction, branding other elements in the process of the construction of theories as psychologism and ignoring abduction in its original form he comes into terms with Einstein's view (see e.g., quote of Einstein in Subsection 5.5.1 and elsewhere in Section 5.5).

The Einstein-Popper view disregards Π, which is replaced by "free invention" and put outside science, this is, outside logical examination. As free invention, the replacement of a concept in a formula by another one must be admitted, although its immediate consequence is that the new theory must be put to test from scratch (see [p. 63, Popper, 1959]). Popper does not address how we go from glyphs into experiments, he apparently ignores Γ as well. Einstein instead introduces "intuition" as part of the assessment of the "truth content". This intuition shall not be confused with Husserl's eidetic intuition that goes from the observed to the facts, for this one moves in the opposite direction, from theory/ideas into observations. In any case, to restore part of the coherence of the old science, such epistemology needs to be complemented with some principles such as the (intuited) relativity principle and the requirement of reproducibility of experiments.

It must be noticed that the absence of logical conditions for the interpretation, as those that emerge from the abduction, makes free interpretation possible as well. The theories resulting from Popper's epistemology ought to be considered less simple (in his terms) than those supported by the present ("pragmaticist") approach since they can elude refutations by changing the interpretation, "saving" the core of mathematical relations.

3.4. Farewell reason, welcome logic

Another distinctive element of the transition was the abandonment of reason in favour of logic. While the philosophers considered themselves as masters in reasoning, the end of the XIXth Century and beginning of the XX witnessed how logicians claimed the quality control of thoughts. We have already discussed how Russell was unable to grasp Kant and Leibniz with respect to space, Carnap applies his analysis of language to Hegel and Heidegger to show that, to him, these philosophers use pseudo statements [Carnap, 1959]. Carnap's overvaluation of language resembles too much the Wittgenstein (disciple of Russell) of the "Tractatus logico-philosophicus" (which the mature Wittgenstein of "Philosophische Untersuchungen" criticised). Carnap attacks the idea of "thing-in-itself", as Mach had previously done on the grounds of being metaphysical. In his criticism of "abstraction" Mach writes:

> In mentally separating a body from the changeable environment in which it moves, what we really do is to extricate a group of sensations on which our thoughts are fastened and which is of relatively greater stability than the others, from the stream of all our sensations. Absolutely unalterable this group is not. Now this, now that member of it appears and disappears, or is altered. In its full identity it never recurs. Yet the sum of its constant elements as compared with the

> sum of its changeable ones, especially if we consider the continuous character of the transition, is always so great that for the purpose in hand the former usually appear sufficient to determine the body's identity. But because we can separate from the group every single member without the body's ceasing to be for us the same, we are easily led to believe that after abstracting all the members something additional would remain. It thus comes to pass that we form the notion of a substance distinct from its attributes, of a thing-in-itself, whilst our sensations are regarded merely as symbols or indications of the properties of this thing-in-itself. But it would be much better to say that bodies or things are compendious mental symbols for groups of sensations—symbols that do not exist outside of thought. Thus, the merchant regards the labels of his boxes merely as indexes of their contents, and not the contrary. He invests their contents, not their labels, with real value. The same economy which induces us to analyse a group and to establish special signs for its component parts, parts which also go to make up other groups, may likewise induce us to mark out by some single symbol a whole group. [The economical nature of physical inquiry, Mach, 2012]

Curiously enough, Hegel had criticised the metaphysical thing-in-itself in his Logic [§ 28, 40, 57, 82, ..., Hegel, 2001a]. In identical form in which Newton is endowed with "absolute space" and then criticised, Hegel is endowed with the "thing-in-itself" and then criticised.[40]

Abstraction will be treated in Chapter 6, let us say here that Mach's view of abstraction is naive as it requires the identification of abstract properties first explicit and later removed, but how were the abstract properties known if not by abstraction from observations? Properties are attributes of objects, not objects in themselves, thus there is no form of applying Mach's concept of "abstract" to the properties. This criticism of Mach is in fact a particular case of the general criticism against the claim that logic is the only form of checking thought quality. It simply consists in showing the lack of self-consciousness in them. Peirce discussed the claim as follows:

> I wish to show the reader how I am going to establish the doctrine of reason in making use of such imperfect reasoning powers as every reader will bring to the subject, and yet without any inadmissible assumption. What a proposition it is that those who pronounce this to be impossible virtually maintain! That imperfect reason cannot perfect itself! How then, I should be glad to be informed, did man ever pass from a state of monership, and lower, to that of lordship of this

[40]Once again we are left with the impression that what Mach (or Carnap for the case) do not understand is labeled as "nonsense", which would be correct if in addition we accept: What Mach (Carnap) does not understand, nobody does.

> globe? Oh, but it was not by a logical process. How so? It was by unwarrantable assumptions. What pedantry! So a man is not to be allowed to see what is reasonable unless he do so by the rules of art! {*To sabbaton dia ton anthropon egeneto, ouch ho anthropos dia to sabbaton*}, which being translated means, "Logic came about for the sake of reasonableness, not reasonableness for the sake of logic." Let us never lose sight of that truth, forgotten though it is, every day, in every walk of life, especially in well-regulated America![CP 2.195, Peirce, 1994]

3.4.1. Criticism of Mach's concept of mass. The problem with abstraction has practical effects. For example, for Newton "mass" is a concept associated to the quantitative outcome of experiments, it is Nature, not our mind, what makes whatever needs to be done to output a value (or a mark on a scale). The meaning of mass does not come from the properties we have associated to the object, Nature completely ignores our thoughts. Yet, Mach searches for the concept in his previous thoughts:

> With regard to the concept of "mass" it is to be observed that the formulation of Newton, which defines mass to be the quantity of matter of a body as measured by the product of its volume and density, is unfortunate. As we can only define density as the mass of unit of volume, the circle is manifest. Newton felt distinctly that in every body there was inherent a property whereby the amount of its motion was determined and perceived that this must be different from weight. He called it, as we still do, mass but he did not succeed in correctly stating this perception. We shall revert later on to this point, and shall stop here only to make the following preliminary remarks.[p. 194, Mach, 1919]
>
> In the first place we do not find the expression "quantity of matter" adapted to explain and elucidate the concept of mass, since that expression itself is not possessed of the requisite clearness. And this is so, though we go back, as many authors have done, to an enumeration of the hypothetical atoms. We only complicate, in so doing, indefensible conceptions. If we place together a number of equal, chemically homogeneous bodies, we can, it may be granted, connect some clear idea with "quantity of matter", and we perceive, also, that the resistance the bodies offer to motion increases with this quantity. But the moment we suppose chemical heterogeneity, the assumption that there is still something that is measurable by the same standard, which something we call quantity of matter, may be suggested by mechanical experiences, but is an

> assumption nevertheless that needs to be justified. [...] [pp. 216–217, Mach, 1919]

Mach's criticism of Newton in terms of circularity is false. As we have shown in Subsection 1.A.3 not only the "quantity of matter" can be measured but there is redundancy as well. The redundancy permits to check the hypothesis regarding the addition of causes made. Mach operates as a logician whose concepts are detached from the conceptualisation (see note 1.A.4). Is density the result of a definition of mass per unit volume? Certainly this is not Archimedes idea, for he new nothing about mass. It is neither Galileo's idea or what his balance measures, he did not know about mass. Mach asserts that history does not make precedence. But, how possibly he arrived to the idea of density not knowing what mass is? We conjecture that he got the idea in class. That he was instructed through a didactic transposition that facilitated the acceptance of beliefs but detached the concept from conceptualisation. A minor price in terms of the formation of professionals, but a crime in the human formation of a scientist. Later in this chapter we will come back to the false pretensions of logical empiricism. For Mach

> Knowledge which is historically first, is not necessarily the foundation of all that is subsequently gained. As more and more facts are discovered and classified, entirely new ideas of general scope can be formed. We have no means of knowing, as yet, which of the physical phenomena goes *deepest*, whether the mechanical phenomena are perhaps not the most superficial of all, or whether all do not go *equally deep*. Even in mechanics we no longer regard the oldest law, the law of the lever, as the foundation of all the other principles.[pp. 495–496, Mach, 1919]

The historical view assures that concepts are introduced after proper conceptualisation and forming a unit with it. However, as we have already mentioned (see Footnote 26 in Chapter 1) Mach prefers to organise knowledge according to "economy of thought" bearing in mind that

> When we wish to bring to the knowledge of a person any phenomena or processes of nature, we have the choice of two methods: we may allow the person to observe matters for himself, when instruction comes to an end; or, we may describe to him the phenomena in some way, so as to save him the trouble of personally making anew each experiment.[p. 5, Mach, 1919]

Experimental knowledge is then conceived as a product that can be stored in books for later use and somehow is shared by instruction. Instruction (indoctrination) precedes experience determining what we have to see in the experience, thus making impossible the phenomenological attitude named *epojé*. Instruction prevents not only the phenomenological attitude but, perhaps more importantly for the State, it prevents independent thinking, Enlightenment and the development of a critical reason.

In Mach the concept of mass (inertial mass, in his terms) is a property of the body that needs not to exist a priori except perhaps as the result of aggregation of “chemically homogeneous bodies”. But this conjectural property (mass) must be empirically demonstrated to be independent of other known properties. In other words, the concept of mass is reached by making abstraction (omission) of all other properties of the bodies, very much in line with his idea of abstraction. In contrast, in Newton it is the result of the synthesis $m = \rho V$. Mach’s views are mostly analytic, he states his conditions for “explanation”

> [...] When once have reached the point where we are everywhere able to detect the same few simple elements, combining in the ordinary manner, then they appear to us as things that are familiar; we are no longer surprised, there is nothing new or strange to us in the phenomena, we feel at home with them, they no longer perplex us, they are *explained*. [pp. 5–6, Mach, 1919]

The phenomena is then explained in a purely analytical form, components are separated and glued together in mind to explain the observed. The unknown is explained in terms of the already known, a procedure that leads to an infinite regression that needs to come to an end by the adoption of beliefs (often named Axioms), which are excepted from the analytical procedure. We can make for example: all correct reasonings conform to logic, except this one that you ought to accept by some ~~other reason~~ irrational form.

Mach’s discussion is likely to appeal to the mind of the adult that has reached the formal stage of development, yet in case they have produced post-formal development it might immediately be apparent that at the end all rests on irrational components, or perhaps on authority. In this respect, it is interesting to observe that along the same process of change of the epistemic practice emerged elitism and the scientists claimed to be the exclusive authority on their parcel of science, for science was parcelled and disciplined in the same process. If we regard reasoning a possibility accessible to humans, we may claim that every person could understand science, conditioned to their innate, developed capabilities and willingness. In contrast, the new science of the bourgeoisie was privatised, claimed to be “owned” by specialists and inaccessible to others. Even worse than privatising science is the accompanying action of derogating reason performed by positivists and instrumentalists:

> By and large, Dirac shared the positivist and instrumentalist attitude of the Copenhagen-Göttingen camp, including its belief that quantum mechanics is devoid of ontological content. He thought that the value of quantum mechanics lay solely in supplying a consistent mathematical scheme that would allow physicists to calculate measurable quantities. This, he claimed, is what physics is about; apart from this, the discipline has no meaning. [p. 80, Kragh, 1990]

"Farewell to reason" is actually a title and thesis by Feyerabend [1987]. The elimination of critical reason and the reduction to instrumental reason was also the subject of [Horkheimer, 1947], and it has repeatedly been reported by other authors such as for example [Sebreli, 2011]. The connection between instrumental reason and domination of nature has been explored as well [Galafassi, 2004].

3.5. On Mach's empiricism

Mach's mechanics presents science as regarded from the epistemological frame of his epoch. Mach cannot find any reason for science other than interest (in the restricted sense of obtaining benefits from it). This position antagonises with von Humboldt's view, it belongs to the motives that degrade human beings. As explained above, it is the economy of thought what in Mach organises science, a concept that must prevail above all others. The philosophical roots of this mandate are not mentioned but we can be certain that he performed introspection and found such ideas acceptable and not repelling as Humboldt did.

Science must be shared through instruction and instruction must precede meeting the phenomena as already explained. History may be suppressed at convenience to improve the economy of knowledge and, when such procedure meets with a contradiction, the criticism must be directed to the people that first thought the problem, never as self-criticism since self-criticism of the students unavoidable implies criticism of the authority of the master that inculcate their beliefs.

Mach appears in front of our eyes as a student trying to come into terms with what he is: the product of guidance and instruction. He indulges into thinking that "intuitive knowledge" is somehow pure and not subjective. We read:

> How does instinctive knowledge originate and what are its contents? Everything which we observe in nature imprints itself uncomprehended and *unanalysed* in our percepts and ideas, which, then, in their turn, mimic the processes of nature in their most general and most striking features. In these accumulated experiences we possess a treasure-store which is ever close at hand and of which only the smallest portion is embodied in clear articulate thought. The circumstance that is far easier to resort to these experiences than is to nature herself, and that are notwithstanding this, free, in the sense indicated, from all subjectivity, invests them with a high value.[p. 28, Mach, 1919]
>
> As already seen, instinctive knowledge enjoys our exceptional confidence. No longer knowing *how* we have acquired it, we cannot criticise the logic by which it was inferred. We have personally contributed nothing to its production. It confronts us with a force and irresistibleness foreign to the products of voluntary experience. It appears to us as something free from subjectivity, and extraneous to us, although we have it constantly at hand so that it is more ours than are the individual facts of nature. [p. 83, Mach, 1919]

We do not know how such a view was generated and/or what reason allows him to be bold with respect to it. Once again, it is necessary for it to be in accordance with the results of his introspection. But it turns to be that the view of "logical empiricism" has been experimentally demonstrated to be incorrect by Piaget and his followers. The idea presented by Mach regarding the Truth value of facts has been named by Husserl as "simple ideation" and contrasts with "philosophical ideation" [Husserl, 1983]. Thus, the enunciation by Mach of the idea "instinctive knowledge" reveals no philosophical ideation.

The replacement of observation by memory, as if what experience is, wouldn't be just in half experienced (using Goethe's words), represents a point of contact with Hume for whom ideas are just a weaker form of sensations [Ch. 1, part 1, secc. 1, Hume, 2011]. But the economical procedure described by Mach (call it intellectual laziness if you want) prevents us from confronting the "facts" (which Mach admits are abstracted observations [p. 482, Mach, 1919]) with true observations. This is possible because we have cancelled our doubts thanks to the dogma of "instinctive knowledge". Mach's problem is found repeatedly in other empiricists and has its root in the lack of self-consciousness regarding our intervention in cognition.

Replacement of observations by memory content may have vast consequences. Suppose we save in memory, associated to motion, only visual images of spatial relations. Would we not have a problem understanding true motion that requires to incorporate a memory of muscular effort? But even worse, if we become habituated to thinking in terms of memories, would we not turn to fantasies (imagination) when true memories are scarce or impossible? For in terms of thoughts, fantasies look like real ideas, except that they lack the process of conceptualisation that relates IW to SW and as such often contain the real and the unreal on an equal footing. After all, what would be the difference when ideas of the real lack as well the process of conceptualisation, the phenomenological connection? The irruption of fantasies in science occurs in such circumstances and the quest against metaphysics ends up opening the door to a new form of metaphysics with its ethers.

It must be emphasised that the root of the process lies in the degradation of science by interest. The price we pay is high: we became enemies of Nature, of which we only think in terms of domination. We lose in the scientist most of the phenomenological capability and critical thought and at the end, we endorse the idea that all life is organised in terms of interest. Our gains have been centred in our success in developing technologies and productivity tools. Thus, the bourgeois science has been successful, it is only that success –the satisfaction of our desires– is not the same as good.

In medieval times all matters could be discussed in their own terms as well as in terms of religion; during the Enlightenment all matters could be considered in their own terms and under the scrutiny of reason; since the rising to power of the bourgeoisie all matters can be seen in their own terms and in terms of profit. Science cannot be an exception.

3.5.1. Relevance of Mach thoughts. Let us make explicit the relevance of Mach's thoughts, since otherwise it would appear that we have picked his writings either at random or with some animosity.

Mach ideas are representative of his time. In his prologue to the first German edition of the Mechanics (dated Prague, 1883), we learn that some of his ideas about mechanics as the centrality of conservation of energy were exposed by Birkhoff (1872) and Helmholtz (1879) as well. His ideas about the "economy of thought" he indicates appeared first (1872) in the pamphlet (his word) "*Die Gestalten der Flüssigkeit*" (The forms of fluids) and in a more detailed form by 1882 in his thesis "*Die ökonomische Natur der physikalischen Forschung*" (The economic nature of physical research). In the mean time Avenarius (1876) had written similar ideas in "*Philosophie als Denken der Welt gemäß dem Princip des kleinsten Kraftmaßes*" (Philosophy as thinking the world according to the principle of the smallest measure of force). The background of this intellectual activity is the consolidation of the "2nd Reich" declared by 1871 after Germany's unification, a period of fast expansion of the empire that concludes in World War I.

The influence of Mach lasts up to our days, especially because his relationism is perceived as related to Einstein's special relativity (a much discussed issue). Less discussed is that in separate comments, Mach lays out a reorganisation of science. We read

- Science is of social interest because it results in economical contributions to society (the bottom line of the economy of thought).
- Knowledge becomes science when transmitted. (Introduction, item 4)
- Scientists form a special class (or guild) of professionals:

  > The transition from this stage to the classified, scientific knowledge and apprehension of facts, first becomes possible on the rise of special classes and professions who make the satisfaction of definite social wants their lifelong vocation. A class of this sort occupies itself with particular kinds of natural processes. The individuals of the class change; old members drop out, and new ones come in. Thus arises a need of imparting to those who are newly come in, the stock of experience and knowledge already possessed; a need of acquainting them with the conditions of the attainment of a definite end so that the result may be determined beforehand. The communication of knowledge is thus the first occasion that compels distinct reflection, as everybody can still observe in himself. Further, that which the old members of a guild mechanically pursue, strikes a new member as unusual and strange, and thus an impulse is given to fresh reflection and investigation. [Introduction]

- Division of labour is advocated.

> The division of labor, the restriction of individual inquirers to limited provinces, the investigation of those provinces as a life-work, are the fundamental conditions of a fruitful development of science. Only by such specialization and restriction of work can the economical instruments of thought requisite for the mastery of a special field be perfected. [p. 505]

It must be said that Mach is aware of some of the dangers that specialisation implies:

> But just here lies a danger— the danger of our overestimating the instruments, with which we are so constantly employed, or even of regarding them as the objective point of science.
>
> 2. Now, such a state of affairs has, in our opinion, actually been produced by the disproportionate formal development of physics. The majority of natural inquirers ascribe to the intellectual implements of physics, to the concepts mass, force, atom, and so forth, whose sole office is to revive economically arranged experiences, a reality beyond and independent of thought. Not only so, but it has even been held that these forces and masses are the real objects of inquiry, and, if once they were fully explored, all the rest would follow from the equilibrium and motion of these masses. A person who knew the world only through the theater, if brought behind the scenes and permitted to view the mechanism of the stage's action, might possibly believe that the real world also was in need of a machine-room, and that if this were once thoroughly explored, we should know all. Similarly, we, too, should beware lest the *intellectual* machinery, employed in the representation of the world on the *stage of thought*, be regarded as the basis of the real world. [p. 505]

- As already explained, Mach advocates a didactic transposition favouring efficiency in instruction.

In this case he does not realise that such attitude might destroy the construction of concepts as it is evidenced by his ahistorical criticism of Newton's introduction of the concept of mass. There is a conflict between "economy of thought" and his ideal relation between the intellectual machinery and the physics of the real world.

3.6. Maxwell and the propagation of light

Since the early times of optical science, the belief that light travels from source to detector at a finite velocity took form. The idea has its roots in mechanical motion and suggested first the existence of a propagation medium and later that of a "particle" named photon (sort of carrying the light to the target). This idea contrasts with e.g., Newtonian gravity, which is considered to plainly exist without further specification.

While the quoted paragraph from Maxwell (see Section 3.2) supports the idea of the ether, Maxwell and also Faraday entertained doubts in this respect. The context of the quoted paragraph is to claim the right to investigate this physical hypothesis, strongly opposed by the Göttingen school that he much admired. Maxwell wrote for and against the Göttingen approach. In what can be considered the most balanced expression, he writes

> That theories apparently so fundamentally opposed should have so large a field of truth common to both is a fact the philosophical importance of which we cannot fully appreciate till we have reached a scientific altitude from which the true relation between hypotheses so different can be seen.[p. 228, Maxwell, 2003] (Address to the Mathematical and Physical Sections of the British Association. (Liverpool, September 15, 1870.))

To his disappointment, the existence of the ether was refuted by experiments while the relational theories were abandoned by adherence to substantialism.

The body-like conception of light refers to an epistemological frame different and incompatible with the relational conception. A neat example of the relevance of the epistemological frame is Michelson-Morley's experiment, that only makes sense if performed for electric disturbances travelling through the ether, opening for questions about velocities with respect to it. Within a relational view, this experiment makes little sense, since source and detector are at relative rest.

It is important to notice that Maxwell's analogies using the ether produced some lasting contributions, like the "displacement current". However, the displacement current can be produced without invoking the ether through the strategy of the Göttingen school: if there is evidence that electromagnetic disturbances propagate as waves, write it in formulae, i.e., build the corresponding phenomenological map. In contrast, Maxwell's expression for the electromechanical force was rejected by experiments. The currently accepted electromagnetic force was produced by Lorentz arguing from the ether, but with a curiosity: the "virtual displacements" of the probe he used were at the same time displacements with respect to the ether and with respect to the remaining electromagnetic objects [§ 71, Lorentz, 1892]. Hence, his argument holds true not because of the ether (call it free thinking) but because it once again agrees with the relational point of view. The details will be addressed in Chapter 4.

3.7. Abduction, analogies, electromagnetism and the ether

The history of electromagnetism is a very rich subject to learn about the epistemological changes in science at the end of the XIX and beginning of the XXth Century. As electromagnetism, EM, posed higher difficulties than mechanics for understanding, scientists began to rest partially on analogies to produce conjectures to be tested by experiments first, and later to promote the integration of EM laws. It is in this context that the idea of the ether irrupts and that finally the idea of fields was established. The process has been studied by Nersessian [1984]. As we have explained and can be read in detail in [Nersessian, 1984] Faraday and Maxwell constantly indicated the advantages as well as the risks entailed in the use of analogies. They are good for suggesting conjectures, but the truth value of them must be established by other methods. Let us analyse from the present perspective Maxwell's proposal.

To construct his set of equations Maxwell took into account experimental results, the principle of minimal action borrowed from mechanics and finally, the conjecture regarding the existence of the ether. Thus, the image of the observed by Π is $\{M, H\}$, where M represents the idealised experiments plus the mechanical principle, and H stands for hypothesis regarding the existence of the ether. The result of his mathematical elaboration, ϕ, was a set of equations labelled from A to E [[619], Maxwell, 1873]. The ether intervened by analogy in the production of the equation for the electromotive intensity (B), the mechanical force (C) and the equation for electric currents (E). We have then a set of intermediate conclusions given by $\{A, \ldots, E\}$. Further elaboration from (B) and (E) (paragraph [783]) leads to the propagation of light as waves. Let W be the statement about the existence of waves and F the mechanical force. Then, we have the implications

$$\{M, H\} \Rightarrow \{A, \ldots, E\} \Rightarrow \{W, F\}$$

Comparison with experimental results obtained interpreting the equations (application of Γ) supports the idea of W being true, but the evidence is opposed to F. Thus, F evaluates to False and the theory must be considered refuted: $\{W, F\}$ evaluates to False. Ignoring the intermediate steps, logic requires that we are in one of three possible cases: $\{M, H\}$ evaluates to {False,True}, or {True, False} or {False, False}. Then, it is not possible to conclude that M is False.

Since later H was found to be False, consideration may move to $\{M, H^*\}$, using an alternative hypothesis H^*. Such an alternative (ether free) hypothesis had been advanced by Gauss [bd.5 p. 627-629, 1870] in a letter to Weber of 1845, and was worked out by Lorenz [1867] to produce the same theory for EM waves as Maxwell's. Later, Lorentz [1892] perfected Maxwell's mechanical hypothesis, changing M by M^* obtaining what is today known as the Lorentz force. Hence, there is a possible EM based upon $\{M^*, H^*\}$ which has not been explored[41]. The historical course was to drop $\{M, H\}$ entirely and to select as starting points some, but not all, of the equations in $\{A, \ldots, E\}$, later adding the Lorentz force. In this form, the phenomenological work by Maxwell was left behind. Speaking of the field equations, Nersessian [p. 20, 2008] says

> ...the field equations will not map back onto the domain from which the models through which Maxwell derived the equations were constructed.

On the other hand, for Nersessian [p. 12, 2008],

> Creative inference is often labeled "abduction", but the nature of the inferential processes of abductive reasoning remains

[41]We claim that the resulting theory differs only in some interpretation from the currently accepted theory but it is more *consilient* [p. XXXIX, aphorism XIV, Whewell, 1840], meaning that it explains a larger class of facts.

> largely unspecified. Formulating an account of model-based reasoning provides a means of specifying the nature of the ampliative reasoning in abductive inference, for instance, analogy.

Notice that "Creative inference" is only a part of the abduction process as understood in the present work, but agrees with the use of "abduction" by other authors. If we have a problem A with properties P_A and a second problem B, not clearly related, with properties $P_B \subset P_A$, then we have the right to guess that perhaps B has the properties in $P_A - P_B$ as well. After checking which $p \in (P_A - P_B)$ are True, we can enlarge the set of properties of B, but unless every possible property that can be tested evaluates to True, the necessary conditions for equivalence of problem B with problem A are absent. Let P_B^* be the augmented set of properties after the test. We might then decide that P_B^* is a good set of axioms for our goals, perhaps with some additional assumptions/analogies that will be treated by iterating the procedure. The result is a patchwork that retains the truth content filtering out the expectations proved to be false. Such process goes against the unity of] science and requires to leave the map Π outside rational scrutiny. This approach is adopted by e.g., the Einstein-Popper epistemology (Section 3.2) and it is the result of the epistemological shift developed during the second half of the XIXth Century.

For a mind inclined to logic the sequence $A \Rightarrow B \Rightarrow \sim A$ simply speaks of inconsistency. Negating the expression we obtain $\sim A \Leftarrow (\sim B) \Leftarrow A$ thus both A and its consequences, B can be thought of as True and False at the same time. The set of premises A deserves to be called inconsistent. This is what happens with the derivation of electromagnetism using mechanical methods and the hypothesis of the ether: they cannot be made to stand together. We speculate that perhaps this problem lead Hertz to his bold remark opening the first of his two theoretical papers:

> The system of ideas and formulae by which Maxwell represented electromagnetic phenomena is in its possible developments richer and more comprehensive than any other of the systems which have been devised for the same purpose. It is certainly desirable that a system which is so perfect, as far as its contents are concerned, should also be perfected as far as possible in regard to its form. The system should be so constructed as to allow its logical foundations to be easily recognised; all unessential ideas should be removed from it, and the relations of the essential ideas should be reduced to their simplest form. In this respect Maxwell's own representation does not indicate the highest attainable goal; it frequently wavers between the conceptions which Maxwell found in existence, and those at which he arrived.[p. 195, Hertz, 1893]

Hertz decided then to sanitise the theory eliminating the potentials and, what is more important, modifying the equations to adapt them to the requisites of his understanding in terms of the *Bild* concept (see Section 3.2). The ether was a necessity of his form of understanding, the bodies have to drag the ether and body-like Galilean invariance was required, and then imposed, modifying the equations.[42]

This is the form of the ether that, soon after, produced wrong insights and was proclaimed failed, we read:

> That transparent bodies can move, without communicating their full velocity to the contained aether, was proven by Fizeau's famous interference experiment with streaming water. [p. 1, Lorentz, 1895]

Are fields really in opposition with mechanics? For the Göttingen school and especially Ludwig Lorenz, the ether was never needed and fields were just a mathematical convenience (when appropriate). For them there is no space and therefore no need to discuss its properties. Only spatial relations are relevant.

For Hertz and many others that relied on analogies for understanding there was no other solution than reifying the space and dressing it with various fields. When Einstein announces in 1905 that the ether is not needed he is actually saying that the properties attributed to the ether can be directly attributed to the space.

In this sense, for Einstein the space is the ether:

> When we speak here of aether, we are, of course, not referring to the corporeal aether of mechanical wave-theory that underlies Newtonian mechanics, whose individual points each have a velocity assigned to them. This theoretical construct has, in my opinion, been superseded by the special theory of relativity. Rather the discussion concerns, much more generally, those things thought of as physically real which, besides ponderable matter consisting of electrical elementary particles, play a role in the causal nexus of physics. Instead of 'aether', one could equally well speak of 'the physical qualities of space'. [...]
>
> It is usually believed that aether is foreign to Newtonian physics and that it was only the wave theory of light which introduced the notion of an omnipresent medium influencing, and affected by, physical phenomena. But this is not the case. Newtonian mechanics had its 'aether' in the sense indicated, albeit under the name 'absolute space'.[Einstein, 1924]

It has been explained in several opportunities [Thomson, 1884, Lange, 1886, DiSalle, 1990, 2020, Solari and Natiello, 2021] that the idea of Newton's mechanics resting on absolute space is alien to Newton. Mechanics rest upon "True motion" which is *not* motion in absolute space.

Synthesising our position: those artefacts needed to organise our thoughts belong with metaphysics. Analogies and imagination are great resources but it

[42] Notice that since we are speaking of actions and not bodies, we have to ask first for the form in which the subjective view of the space must be removed in this case rather than acting by analogy. This is, Hertz motivation is correct but the resolution requires a cognitive surpass –and not analogy.

is necessary to use them with the caution advised by Faraday and Maxwell, for otherwise they propitiate metaphysical artefacts.

Appendix

3.A. Scholium. On Maxwell's argument against the Göttingen theory

In a communication to the Royal Society read June 18, 1868 [pp. 125–143, Maxwell, 2003], Maxwell comments and criticises the electromagnetic theories of Lorenz and Riemann. He would later in practice retract from this criticism in his treatise [Maxwell, 1873], admitting that Lorenz' equations are equivalent to his system and, as we have seen in Section 3.6, also in a communication addressed to the Mathematical and Physical Sections of the British Association (Liverpool, September 15, 1870) [pp. 215–229, Maxwell, 2003]. Nevertheless, it is interesting to use his argument to show how interpretation and analogy irrupts in scientific argumentation even in the words of a celebrated scientist. Maxwell writes [pp. 137–138, Maxwell, 2003]:

> **For let two oppositely electrified bodies A and B travel along the line joining them with equal velocities in the direction AB, then if either the potential or the attraction of the bodies at a given time is that due to their position at some former time (as these authors suppose), B, the foremost body, will attract A forwards more than A attracts B backward.** Now let A and B be kept asunder by a rigid rod. The combined system, if set in motion in the direction AB, will pull in that direction with a force which may either continually augment the velocity, or may be used as an inexhaustible source of energy. I think that these remarkable deductions from the latest developments of Weber and Neumann's theory can only be avoided by recognizing the action of a medium in electrical phenomena. (emphasis added)

On a technical aspect, the "combined system" can be described easily with Lagrange's formulation. It results in a Lagrangian independent of time (despite the retarded interaction, which is nullified by the rigid rod), meaning that Hamilton's function is constant in time. With the introduction of generalised coordinates it does not require an explicit dependence on time. Hamilton's function is the energy. The system does not create energy. Furthermore, since the interaction he considered was associated exclusively to Coulomb's potential, the acceleration of the system is strictly zero and the potential is constant along the trajectory. The matter illustrates the dangers in "talking physics".

The highlighted part deserves further analysis. Assume the described situation to be possible without coercing it through a rod. The electrical action jumps to the space of the emitter and it delivers the action only when it is at a place in coincidence with the receiver (it sounds pretty much like a modern photon emitted from A towards B). Let t_B be the time when the signal that left A is sensed by B and t_0 the time of emission. The distance travelled by this signal would be then $X_B(t_B) - X_A(t_0)$ (as measured in the frame of A, a subjective distance –see Subsection 1.2.5–). This coincides with the distance travelled by a body moving in a straight line from $X_A(t_0)$ to $X_B(t_B)$. In the same form, let t_A be the time when the signal that left B at t_0 is sensed by A. The corresponding distance is then $X_A(t_A) - X_B(t_0)$ (in the frame of B). But since B is moving along the straight line AB while the signal is travelling, we have $|X_B(t_B) - X_A(t_0)| > |X_A(t_A) - X_B(t_0)|$. This relationship between positions, which is obvious for material bodies, is hence assumed to hold for electromagnetic interactions as well. We must admit that this form of thinking is odd: how do we know the signal jumps to the space of the receiver at the end of the process and not at the beginning? More relevant: this interpretation breaks the original objective conception of the Göttingen school in terms of relative positions and velocities. The relative distance between A and B is the same as that between B and A, except for orientation. We have broken the contract, and eq. (2.3.1) is not satisfied, a tribute to the use of intuitions based upon images by analogy.

In any case, the observer that will measure the electromagnetic signal is associated to the detector and not to the emitter. Let us say that an array of detectors is placed in the frame of the receiver to track the electrical action. The receiver (observer) will register the distance between source and target at the time of emission. The action is tracked by the detectors of any observer since the moment of emission and on this basis we can compute the velocity of the signal. If τ is the time taken to reach the detector, we have $X_B(t_0) - X_A(t_0)$ which is an objective distance (equally registered from any reference system), as well as τ, both invariant in classical physics. Then, $\dfrac{X_B(t_0) - X_A(t_0)}{\tau}$ is an invariant velocity and there is no problem involved. All observers determine the same value. The price paid is to leave analogies behind.

The "travelled distance" is a concept pertaining to absolute space. In contrast, we usually measure distances travelled by vehicles using an odograph. Place a car in a freight train and deliver it to some distant place. The odograph of the train will indicate a distance between train stops and the odograph of the car will indicate zero. We know both readings are correct, distances travelled are relative as well as velocities. To pretend that electromagnetic action has a speed solely determined by its identity is the same that asserting that light moves in absolute space.

CHAPTER 4

The Electromagnetism of the Göttingen school.

Our intention in this chapter is not to show how electromagnetism came about, much less how its current formulation reads, although we make contact with it. We assume that the reader is acquainted with an undergraduate course of electromagnetism. We want to highlight first some historical events that are not part of the "official historic narrative" regarding the making of electromagnetism as a mature theory. We intend to show how the epistemic praxis adopted by the end of the XIXth Century impacted in the construction of physics, how the introduction of the ether facilitated acceptance of Maxwell's theory but obscured physics and how the old epistemic praxis solves the challenges presented by the experiments.

4.1. Notes on the history of Electromagnetism

No history has a beginning and electromagnetism is not an exception. We intend only to comment some findings that had an important impact in the evolution of our understanding of this matter.

The Leyden jar (1745). In October 1745 Georg von Kleist devised a means to store electricity. A metal nail was dipped in water inside a glass bottle, held by an assistant standing on an insulator, the nail connected to a friction generator. When the assistant touched the nail with the spare hand, a powerful discharge took place. The news of the phenomenon arrived to Pieter van Musschenbroek from Leyden who along with Andreas Cunaeus and Jean Allamand realised that electrical conductors were required (e.g., metal foils inside and outside the bottle), with the bottle acting as an insulator between the conductors and the whole apparatus insulated from the environment. A fairly large quantity of charge of opposite sign accumulated on the conductors. The bottle was the original form of a *capacitor*. Musschenbroek communicated the experiment to the Académie des Sciences in Paris in 1746 [l'Abbé Nollet, 1746]. The Leyden jar provides a stable source of electricity, in this sense far superior to the friction generators that where used to "fill" the jar. It became part of the basic equipment of experimental electricity.

Experiments by Coulomb (1785). Coulomb published his *premier mémoire* (of totally seven) in 1785 [Coulomb, 1785], reporting the construction of an electric (torsion) balance. An horizontal insulated bar was held by a thin vertical

FIGURE 4.1.1. Left: Leyden jar; Right: Volta battery image (Photographs by Rama, Wikimedia Commons, Cc-by-sa-3.0-fr; edited)

silk wire and a metal ball attached to one end of the bar. By charging the ball with a known amount of electricity and approaching another charged ball to it (moving in the horizontal plane), the bar will twist on the horizontal plane until the restoring force from the silk thread (that can be independently determined) exactly compensates this force. In this way he established that two balls with opposite (same) electrostatic charges exert an attracting (repelling) force on each other, proportional to the size of the charges and varying as the inverse square of the relative distance. This was the first attempt to summarise the action of electric charges, and in this sense becomes a model for later developments.

Invention of the voltaic battery by Alessandro Volta (1799). In 1800 Volta reported the construction of a device consisting of layers of copper (or silver) and tin (or zinc), each pair separated by a wet piece of cardboard [Volta, 1800]. A sustained chemical reaction generated electricity, which could be sensed just by touching one copper layer and one of tin, thus operating a discharge similar to those in Leyden jars. The more layers separating the touching points, the stronger the discharge. Volta was inspired by electric rays (cartilaginous fish of the order *Torpediniformes*), having organs arranged in a similar way and capable of producing electric discharges. The voltaic battery allowed for experimentation with sustained currents rather than just sudden discharges. The electricity provided was more powerful and lasting that the Leyden jar, but, was it the same thing? Experiments made with Volta batteries were referred to as "voltaic electricity".

Experiments by Ørsted (1820). A copper wire connected to a voltaic battery is placed nearby a magnetic needle. The needle by itself is oriented towards the magnetic north. When electric current is allowed to flow through the wire (oriented as the needle), the needle deflects towards the east-west direction. The deflection decreases with decreasing power of the battery as well as with increasing distance between needle and wire. Reported on July 21, 1820 [Ørsted, 1820]. Ørsted experiment initiated a new era, until then magnetism and electricity were separate matters and both obeyed Coulomb-like laws, although a magnetic monopole was never isolated. Electrical currents had associated magnetic fields.

The Biot-Savart law (1820). Inspired by Ørsted's experiment, Biot and Savart proposed that a voltaic current passing through a wire magnetises the wire, and demonstrated [p. 222-223, Biot and Savart, 1820] the subsequent magnetic forces for the Académie des Sciences in the autumn of 1820 (at the same time as Ampère and Arago, see below). In modern language, they provided an expression for the magnetic field generated by the current through the wire.

Arago's rotating disc (1824). The experiment is most beautifully described by Faraday [p. 24, § 81, Faraday, 1839]:

> If a plate of copper be revolved close to a magnetic needle, or magnet, suspended in such a way that the latter may rotate in a plane parallel to that of the former, the magnet tends to follow the motion of the plate; or if the magnet be revolved, the plate tends to follow its motion ; and the effect is so powerful, that magnets or plates of many pounds weight may be thus carried round. If the magnet and plate be at rest relative to each other, not the slightest effect, attractive or repulsive, or of any kind, can be observed between them.

This fundamental experiment demonstrates the existence of a force operating when a conductor and a magnet are in relative motion. It triggered Faraday's studies on electromagnetic induction. Arago communicated the experiment to the Académie des Sciences in 1824 [Arago, 1824].[43]All of Arago's experiments illustrate that relative motion between adequate electromagnetic entities generates specific forces, thus altering the state of relative motion. Arago's experiment added a new element, conductors moving under the influence of magnetic fields became magnetised while

[43]Some simple demonstration of Arago's findings can be made in class. If a magnet is suspended making a pendulum with any one of the magnetic poles being the lowest point of the pendulum and the pendulum is made to oscillate above and close to plates of different materials, it will be observed that the amplitude of the oscillations decreases faster when the plate is made of a conducting material (copper, aluminium, iron and some charcoals) than when it is made of isolating materials like wood. The original demonstration is attributed to Arago as well [Arago, 1826]. Also, a piece of magnetised iron sliding along a conducting inclined plane requires a longer time to reach the bottom of the plane than a similar but demagnetised piece of iron.

in motion. Unlike Coulomb forces and magnetic forces, mechanical relative motion entered the puzzle.

Experiments by Ampère (1820-1825). Ampère took notice of Ørsted's experiments through the demonstrations of Arago at the Académie des Sciences in 1820 and subsequently conceived additional experiments and published a series of *Mémoires*, collected in [Ampère, 1823]. He introduced a new setup for Ørsted's experiment (with the needle aligned vertically and the wire perpendicular to it) showing attraction/repulsion phenomena instead of needle deflection. [p. 199-200, Ampère, 1820]

Ampère coined the expressions electrostatic (interactions between electrified bodies at rest), electromagnetic (interactions between current elements and magnets) and electrodynamic (interactions between current elements) and culminated by proposing an expression for the force between two current elements, varying with the inverse square of the distance between elements. He related electrostatic, electromagnetic and electrodynamic interactions, all these forces could be studied and compared to each other by attaining equilibrium conditions. Ampère's experiments gave the laws of mechanical action and together with the Biot-Savart law complete what is known as magnetostatics. The experiments that confirmed that interactions among electric circuits are compatible with Ampère's force had to wait another 25 years until Weber developed the required measurement devices.

Ampère proposed that some currents were responsible for the magnetism in permanent magnets [pp. 183–184, Ampère, 1820] and some ideas trying to explain voltaic currents in terms of the dissociation of molecules and electrical discharges related to the Leyden jars:

> Comme le liquide interposé entre les plaques de la pile est, sans comparaison, moins bon conducteur que le fil métallique qui en joint les extrémités, il se passe un temps, très-court à la vérité, mais cependant appréciable, pendant lequel l'électricité intermoléculaire, supposée d'abord en équilibre, se décompose dans chacun des intervalles compris entre deux molécules de ce fil. Cette décomposition augmente graduellement jusqu'à ce que l'électricité positive d'un intervalle se réunisse à l'électricité négative de l'intervalle qui le suit immédiatement dans le sens du courant, et son électricité négative à l'électricité positive de l'intervalle précedent. Cette réunion ne peut être qu'instantanée comme la décharge d'une bouteille de Leyde; et l'action entre les fils conducteurs, qui se développe, pendant qu'elle a lieu, en sens contraire de celle qu'ils exerçaient lors de la décomposition, ne peut par conséquent diminuer l'effet de celle-ci, car l'effet produit par une force est en raison composée de son intensité. [p. 300, Ampère, 1823][44]

[44]As the liquid interposed between the plates of the battery is, without comparison, less of a conductor than the metal wire which joins its ends, a time passes, very short in truth, but nevertheless appreciable, during which the intermolecular electricity, initially supposed to be in equilibrium, decomposes in each of the gaps between two molecules of this wire.

The paragraph is interesting as it illustrates the mental construction by Ampère. The construction is both completely inaccurate according to current understanding and completely irrelevant for the results obtained by Ampère, it corresponds to a psychological need and is constructed by analogy. In the same form, the magnetic currents in permanent magnets (ferromagnetism) correspond to a guess based upon the idea that electricity surrounds molecules, in this case producing a current [pp. 183–184, Ampère, 1820].

Experiments by Faraday (1831). Faraday's contribution to EM are enormous. We pick as a representative phenomenon the *Faraday disc* or *unipolar generator*, whose behaviour is still being discussed in the scientific literature [Kelly, 2004, Müller, 2014]. It consists of a conducting disc rotating around a perpendicular axis passing through its centre, in the presence of an homogeneous static magnetic field oriented along the rotation axis. A galvanometer is attached to wires touching the rim and centre of the disc, thus displaying the occurrence of a DC current [pp. 24-36, Faraday, 1839]. Faraday's research was aimed to explain Arago's findings, in page 26 he describes the generator.

The device is relevant for the study of induction phenomena, namely when relative motion between electromagnetic objects results in a sustained current. In other words, kinetic energy (from the relative motion) is converted into electromagnetic energy (e.g., the current). Faraday concludes:

> Here therefore was demonstrated the production of a permanent current of electricity by ordinary magnets [§ 90, p. 27, Faraday, 1839].

Faraday's investigations on induction allowed him to propose to measure magnetic fields using the electrical current generated on a moving wire [XXVIII memory, Faraday, 1855]. A relation today known as Faraday's induction law (which can also be stated as: when there is a variation of the magnetic flux generated by a source over the area of a conducting circuit, then a current is induced on the conducting circuit). The problem begins to come together. Conductors in motion in the presence of magnetic fields produce electrical currents and those electrical currents produce the magnetism detected in Arago's experiments.

Wilhelm Weber (1846-57). Wilhelm Weber, friend and disciple of Gauss, was a cornerstone in the development of electromagnetic theory. He conceived novel instrumentation and devised experiments in order to put electromagnetic theory to test. He wrote a series of memories under the general name of *Electrodynamic Measurements*, the first of which [Weber, 1846] described a new measurement apparatus, the electrodynamometer, and confirmed that Ampère's proposed

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This decomposition gradually increases until the positive electricity of one interval joins the negative electricity of the interval immediately following it in the direction of the current, and its negative electricity joins the positive electricity of the preceding interval. This reunion can only be instantaneous, like the discharge of a Leyden bottle; and the action between the conducting wires, which develops, while it is taking place, in the opposite direction to that which they exerted at the time of decomposition, cannot consequently diminish the effect of the latter, because the effect produced by a force is in proportion to its intensity. (Translated by DeepL https://www.deepl.com/es/translator) electrodynamic force between conductors was compatible with the experimental observations. Moreover, he proposed extensions to Coulomb's electrostatic force (the *Weber force*) in order to take into account Ampère's electrodynamic forces and Faraday's induction. He was able to measure the time of discharge of Leyden jars [pp. 207–247, On the measurement of electro-dynamic forces, Assis, 2021a] and to reduce the determination of electric and magnetic forces to space, time and mass measurements, thus providing a standard of comparison. He was the first to measure $\epsilon_0\mu_0$ [Assis, 2003], the product of universal constants appearing in electrostatic and magnetic forces. This product appears in the propagation of light waves according to the relation $\epsilon_0\mu_0 C^2 = 1$, where C is called *velocity of light*. Further, he connected electromagnetic phenomena with waves of velocity C through the *telegraph equation* [p. 267-383, Electrodynamic Measurements, Fifth Memoir, relating specially to Electric Oscillations, Assis, 2021b], independently discovered by Kirchhoff [Kirchhoff, 1857]. In a comment by the editor J. C. Poggendorf on Kirchoff's paper it is mentioned that Weber had by then "a complete treatise on the same subject. [...] [B]oth works, starting from essentially the same basis, have led to identical results." [p. 225-228, Assis, 2021b].

Weber's theoretical developments assumed that electricity consists of positive and negative charges. The accepted macroscopic picture today consists of two types of ions where the velocity entering in Weber's force is that of the relative motion of the ions inside a conductor.

The work by Kohlrausch and Weber was conclusive for the unifying of electricity. The currents provided by Volta's battery and the Leyden jar were equivalent in almost all effects, except for the static charge present in the currents associated to the Leyden jar. Weber determined the current from the totality of its known actions: production of a magnetic field, electrolysis of water, register of the galvanometer (proportional to the current) and of the electrodynamometer (proportional to the square of the current) and electrodynamic forces. Current is then what it is implied by these actions.

Franz Neumann (1847). Franz Neumann derived Faraday's law of induction from Ampère's electrodynamic force, through the introduction of the *vector potential* A. Together with Ohm's law this suggests that the electric field responsible for induction effects is not the electrostatic one $(-\nabla V)$ but $-\dfrac{\partial A}{\partial t}$ [p. 124, Assis, 1994].

Faraday and Lorenz: light as an electromagnetic wave phenomena. By 1860 the unification of electromagnetism was in progress and several independent researchers were active seeking a unified form. The starting point was in some cases Weber's theory that elaborates from Coulomb's theory and action at distance and in other cases Biot-Savart's and Neumann's approaches that treat separately Coulomb's law with action at distance or appeal to the ether. Both Weber's theory and Coulomb+Neumann theories accounted for all the known experiments.

A famous 1845 letter from Gauss to Weber [bd.5 p. 627-629, Gauss, 1870] indicating that changes in electromagnetic configurations influenced distant electromagnetic sensors with some delay was exerting as much influence as experimental results. The idea that light was some kind of electromagnetic effect was present as well, at least since Faraday [p. 1, Faraday, 1855] reported that a magnetic field was able to change the polarisation of light beams.

During 1861 the danish scientist Ludwig Lorenz published two independent papers, the first one regarding the propagation of waves in elastic media. In this paper he introduced the differential equations for waves and their integrals, today known as delayed propagation [Lorenz, 1861b]. In the second paper [Lorenz, 1861a] he described light as transversal vibrations with independent polarisation directions oscillating in correspondence with waves. The work was based both in his experiments and his mathematical knowledge of wave theory. Two years later he published "On the theory of light" (propagation in non-adsorbent heterogeneous media) where he endeavoured to describe all known characteristics of light with "the smallest possible number of hypothetical assumptions..." and found that by further developing the formal (mathematical) part of the theory "an essential part of the ordinary physical hypothesis are not needed for the explanation of the phenomena of light". This is, Lorenz introduces a change in the epistemic praxis that eliminates physical hypothesis. In particular, he gets rid of the ether.

If we contrast epistemic praxes, we notice that Lorenz, unlike Ampère, avoids making a physical model of the space to support the mathematical correspondence between phenomena. Unlike Ampère that follows the dictum "to like effects like causes" introducing physical hypothesis impossible to put to test and to all effects useless since they are present only to facilitate our acceptance of the experimental results, Lorenz understands they are not needed and that it is safer to avoid them. We are again in front of the phenomenological attitude presented by Newton: "*hypothesis non fingo*".

During the decade of 1860/70 two line of thoughts were consolidated. Maxwell's introduction to his decisive work published in 1865 explains it well:

> THE most obvious mechanical phenomenon in electrical and magnetical experiments is the mutual action by which bodies in certain states set each other in motion while still at a sensorial distance from each other. The first step, therefore, in reducing these phenomena into scientific form, is to ascertain the magnitude and direction of the force acting between the bodies, and when it is found that this force depends in a certain way upon the relative position of the bodies and on their electric or magnetic condition, it seems at first sight natural to explain the facts by assuming the existence of something either at rest or in motion in each body, constituting its electric or magnetic state, and capable of acting at a distance according to mathematical laws.
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> In this way mathematical theories of statical electricity, of magnetism, of the mechanical action between conductors

carrying currents, and of the induction of currents have been formed. In these theories the force acting between the two bodies is treated with reference only to the condition of the bodies and their relative position, and without any express consideration of the surrounding medium.

These theories assume, more or less explicitly, the existence of substances the particles of which have the property of acting on one another at a distance by attraction or repulsion. The most complete development of a theory of this kind is that of M. W. WEBER, who has made the same theory include electrostatic and electromagnetic phenomena.

In doing so, however, he has found it necessary to assume that the force between two electric particles depends on their relative velocity, as well as on the distance.

This theory, as developed by MM. W. WEBER and C. NEUMANN is exceedingly ingenious, and wonderfully comprehensive in its application to the phenomena of statical electricity, electromagnetic attractions, induction of currents and diamagnetic phenomena; and it comes to us with the more authority, as it has served to guide the speculations of one who has made so great an advance in the practical part of electric science, both by introducing a consistent system of units in electrical measurement and by actually determining electrical quantities with an accuracy hitherto unknown.

[...] The theory I propose may therefore be called a theory of the *Electromagnetic Field*, because it has to do with the space in the neighbourhood of the electric or magnetic bodies, and it may be called a *Dynamical Theory*, because it assumes that in that space there is matter in motion, by which the observed electromagnetic phenomena are produced.

The electromagnetic field is that part of space which contains and surrounds bodies in electric or magnetic conditions.

It may be filled with any kind of matter, or we may endeavour to render it empty of all gross matter, as in the case of GEISSLER'S tubes and other so-called vacua.

There is always, however, enough of matter left to receive and transmit the undulations of light and heat, and it is because the transmission of these radiations is not greatly altered when transparent bodies of measurable density are substituted for the so-called vacuum, that we are obliged to admit that the undulations are those of an aethereal substance, and not of the gross matter, the presence of which merely modifies in some way the motion of the aether. [pp. 459–460, Maxwell, 1865]

We name **Göttingen school** the approach inspired by Gauss who greatly influenced his friend Weber especially while the latter was a professor at Göttingen[45]. This school refused to introduce "physical hypotheses" like the ether, following Newton's epistemological tradition. When in need to picture the situation Weber describe electricity in matter as "electrical particles" [Weber, 1846] while the action resulting from the electrical particles and their motion was perceived at distance, a fact accepted without need of further explanation. Weber's force not only depended on velocity, it depended on acceleration as well. It was criticised by Helmholtz since forces that depend on velocity cannot preserve the energy, but the criticism is superficial, the theorem regarding the conservation of energy does not consider the dependency with the relative acceleration. Weber showed later that his force preserved energy [Weber, 1872]. It actually corresponds to a Lagrangian that depends on velocities (see [Assis, 1994]). Along these lines worked as well Riemann [1867], Betti [1867], Lorenz [1867], Neumann [1868] and Lorentz (1878) [Lorentz, 1936].

The historical context of the unification of Germany and the emergence of the Prussian Empire (Second *Reich*) might have influenced the development of physics. W. Weber was born in 1804 in Wittenberg (Saxony), that in 1806 became part of the French (Napoleonic) Empire. The town was recovered by Prussia after heavy bombing (1813-1814). The University of Wittenberg was dissolved in 1815 and his father, a professor of theology, moved to Halle. Weber's theories were disputed up to an irrational point by Helmholtz (Berlin U.)[46] as already indicated. The rejection of Weber by Prussia was such that despite that the name "Weber" was in use to denote the unit of current, Helmholtz proposed the name "Ampere" in a Congress in Paris 1881 [p. 51, Assis, 1994]. The work of Carl Neumann endured like polemics with Helmholtz [Schlote, 2004, Jungnickel and McCormmach, 2017]. After Riemann's death (Italy, 1866) a work he had submitted to the "Poggendorff's annalen") by 1858 (withdrawn before publication) was published [Riemann, 1867]. At the same time, Betti (who was in fluent conversation with Riemann in Rieman's last years) published a note [Betti, 1867] explaining which had been the reason for Riemann to withdraw the paper and how Riemann's objection could be removed. Ludwig Lorenz [Lorenz, 1867] published his work as well in 1867, in the same number and exactly next to Riemman's paper. Finally Carl Neumann published his version of delayed action at distance [Neumann, 1869]. After Neumann's work, Rudolf Clausius (a physicist and Prussian patriot that volunteered for the French-Prussian war) published a criticism of Göttingen's electromagnetism under the title "Upon the new Conception of Electrodynamic Phenomena suggested by Gauss" [Clausius, 1869]. Clausius criticism considered Neumann, Riemann and Betti's work (despite Betti's being published in Italian in "Il nuovo cimento") yet he did not mention Lorenz work that was next to Riemann's. Clausius criticism is

[45]W. Weber was one of the "Göttingen seven". Professors that were fired for opposing the derogation of the liberal constitution of the Kingdom of Hannover (1837).

[46]In turn (von) Helmholtz was recognised by Prussia's king with the nobility addition of "von" by 1883.

based in the (implicit) assumption that to explain/understand it is needed to provide a "physical theory". Indeed, the criticism boils down to "I do not understand these gentlemen". The reason Clausius did not include Lorenz in his criticism is not clear, perhaps the reason is the very clear philosophical statement that Lorenz made and prevents the arguments by Clausius from the root (see subsection 4.1). In such case, Gauss conception survives, and it was fully recognised later by Lorentz [Lorentz, 1892]. Let us remember that Göttingen (Hannover) was annexed (by force[47]) to Prussia becoming in 1868 the Province of Hannover. Neumann's attempts to carry on with scientific argumentation were mostly ignored, his latest attempts were made at the beginning of the XXth Century [Jungnickel and McCormmach, 2017]. The period presents other events that we can only explain as philosophical limitations or plain intellectual dishonesty. Helmholtz in his homage to Marcus writes

> [...] With Faraday, the antagonism to the physical theories hitherto held, which treated of atoms and forces acting at a distance, was even more pronounced than with Magnus.[p. 21, Helmholtz, 1908]

which is a gross misrepresentation of Faraday's thoughts and his entertainment of doubt[48]. Helmholtz' influence extended to philosophy where he is considered one of the earliest Neo-kantians [Beiser, 2014], a movement that intended to make the Sciences independent of philosophy, which means to limit the scope of Kant's idea regarding the task of the minor faculty. Reason was no longer above all, the Enlightenment had began to yield to socio-economic pressure. The emergence of socio-political manoeuvres within the philosophical and scientific community is just an indication of the drastic changes that the civilisation was undergoing.

James Clerk Maxwell. Maxwell apportions are better appreciated when exposed extensively in his "Treatise of Electricity and Magnetism" [Maxwell, 1873]. His work follows four main directions:

(1) To extend formulae introduced for wires (assumed to be one dimensional objects) to objects with significant volume.
(2) To systematically examine the relations produced rewriting them applying Gauss theorem and its corollaries.
(3) To propose a Lagrangian form for induction using as starting point F. Neumann's formulation [p. 190 and Ch. VII, Maxwell, 1873].

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[47] Battle of Langensalza, June 27th 1866, during the Austro-Prussian War.

[48] Wise [1983] indicates that Helmholtz in his debate with "main stream vitalism" systematically distorted the ideas of his opponents. He indicates as well that the confrontation between Helmholtz and Weber was the result of the adherence of them to different philosophical stand points, the historicism associated to Goethe and Hegel in the case of Weber and a "radical materialism" in the case of Helmholtz.

Helmholtz' open attacks to the defunct Hegel and his followers are well documented. For example, in "On the relation of natural science to general science" [Helmholtz, 1873] (a talk offered in 1862) he criticises a straw-man version of Hegel's point of view ignoring the work of W. Whewell who had presented ideas on Science that were inspired in Hegel's philosophy (circa 1837 – 1865). Whewell was a very influencing philosopher in Great Britain, Master of the Trinity college (1861-1866).

(4) To propose the "displacement current" [[610], Maxwell, 1873] (the current induced in the ether), proceeding then by analogy to replace in the formulae obtained from experiments the current in the conductor by the total current (conductor plus ether) [[610, 619], Maxwell, 1873]. The sequence of equations (E,F,G,H,I) presented in paragraphs [607-610] are responsible for the propagation of light as a wave.

Following this method he produced a set of nineteen formulas, adding to them the continuity equation (conservation of charge) to conform twenty equations [p. 465, Maxwell, 1865]. Yet, equation (C) in [[603, 619], Maxwell, 1873], that is said to correspond to a mechanical force, is falsified by experience, and therefore the theory is refuted. Since the incorrect term in equation (C) was added by an arbitrary decision (analogy), through the same argument it could be dropped, but this is only possible by admitting the arbitrariness of the construction. Utilitarianism would not object such action. The form in which Maxwell produces his equation (I) [[606-611], Maxwell, 1873], which includes the displacement current, parallels the work by Kirchhoff [Kirchhoff, 1857] later presented by Weber [p. 270-274, Assis, 2021b] (1864, V memory). In Kirchhoff and Weber the displacement current is presented with greater detail than in Maxwell. The only difference is that, since there is no such thing as the ether, it is restricted to where it exists: "inside matter". As indicated by Maxwell [607] referring to displacement currents: "Their importance will be seen when we come to the electromagnetic theory of light". Faraday would not approve since, according to the paragraph quoted in Scholium 2.B, this is an "unsafe" action that could lead to forget the hypothetical character of the ether (because of its success at explaining what Maxwell wanted to explain) as it had happened before. In terms of epistemic practice there is an abyss between Faraday and Maxwell's followers, while Maxwell lies in between the philosopher and the scientists.

Concluding the work, he introduced the Electromagnetic energy and discussed the stress in the ether and the propagation of light as propagation of stress in the ether. Maxwell recognises that his and Lorenz's descriptions of light propagation arrive to similar conclusions, despite the difference in starting points [note in [805], p. 449-450, Maxwell, 1873] and the different epistemic praxis, being Lorenz in consonance with the Göttingen school and Newton.

Maxwell insisted in connecting his mathematics with Faraday's expressions and in such sense he helped to put in value the extraordinary work by Faraday. However, in an essential point he decides to depart from Faraday. Faraday elaborated on the possibility of conceiving matter through its actions rather than as isolated atoms surrounded by space and existing independently of these actions. Maxwell stood closer to this second option. We read in [[529], p. 177, Maxwell, 1873] (about Faraday),

> He even speaks * of the lines of force belonging to a body as in some sense part of itself, so that in its action on distant bodies it cannot be said to act where it is not. This, however, is not a dominant idea with Faraday. I think he would rather have said that the field of space is full of lines of force,

> whose arrangement depends on that of the bodies in the field, and that the mechanical and electrical action on each body is determined by the lines which abut on it.

where the reference * is [p. 293, Faraday, 1844] and [p. 447, Faraday, 1855]. Faraday actually says (elaborating on Boscovich's view of "atoms consisting merely of centres of force" [p. 291, Faraday, 1844]):

> The view now stated of the constitution of matter would seem to involve necessarily the conclusion that matter fills all space [...]. In that view matter is not merely mutually penetrable, but each atom extends, so to say, throughout the whole of the solar system, yet always retaining its own centre of force. This, at first sight, seems to fall in very harmoniously with Mossotti's mathematical investigations and reference of the phenomena of electricity, cohesion, gravitation, &c. to one force in matter ; and also again with the old adage, " matter cannot act where it is not." [p. 293, Faraday, 1844]
>
> The point intended to be set forth for the consideration of the hearers was, whether it was not possible that the vibrations which in a certain theory are assumed to account for radiation and radiant phænomena may not occur in the lines of force which connect particles, and consequently masses of matter together; a notion which, as far as it is admitted, will dispense with the æther, which, in another view is supposed to be the medium in which these vibrations take place.
>
> You are aware of the speculation which I sometime since uttered respecting that view of the nature of matter which considers its ultimate atoms as centres of force, and not as so many little bodies surrounded by forces, the bodies being considered in the abstract as independent of the forces and capable of existing without them. In the latter view, these little particles have a definite form and a certain limited size; in the former view such is not the case, for that which represents size may be considered as extending to any distance to which the lines of force of the particle extend: the particle indeed is supposed to exist only by these forces, and where they are it is. [p. 447, Faraday, 1855]
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> The view which I am so bold as to put forth considers, therefore, radiation as a high species of vibration in the lines of force which are known to connect particles and also masses of matter together. It endeavours to dismiss the æther, but not the vibrations. [...]
>
> The æther is assumed as pervading all bodies as well as space: in the view now set forth, it is the forces of the atomic centres which pervade (and make) all bodies, and also penetrate all space. As regards space, the difference is, that

> the æther presents successive parts or centres of action, and the present supposition only lines of action; as regards matter, the difference is, that the æther lies between the particles and so carries on the vibrations, whilst as respects the supposition, it is by the lines of force between the centres of the particles that the vibration is continued. [p. 451, Faraday, 1855]

These paragraphs help to illustrate how Faraday entertained several possibilities while Maxwell was gained by one of them, yet, both of them were fully aware of the hypothetical character of the discussion. Maxwell closes his book saying:

> Hence all these theories lead to the conception of a medium in which the propagation takes place, and if we admit this medium as an hypothesis, I think it ought to occupy a prominent place in our investigations, and that we ought to endeavour to construct a mental representation of all the details of its action, and this has been my constant aim in this treatise. [[866], p. 493, Maxwell, 1873]

Ludwig Lorenz. As a continuation of his research on light, Lorenz published in 1867 a work "On the identity of the vibrations of light with electrical currents" [Lorenz, 1867]. The paper begins with a declaration that explains his epistemic practice and deserves to be reproduced:

> THE science of our century has succeeded in demonstrating so many relations between the various forces (between electricity and magnetism, between heat, light, molecular and chemical actions), that we are in a sense necessarily led to regard them as *manifestations of one and the same force*, which, according to circumstances, occurs under different forms. But though this has been the guiding idea with the greatest inquirers of our time, it has been by no means theoretically established; and though the connexion between the various forces has been demonstrated, it has only been explained in single points. Thus Ampère has theoretically explained the connexion between electricity and magnetism, though he has not furnished a proof of the possibility of the peculiar molecular electrical currents (assumed by him) which in virtue of their own power are continuous; and, in like manner, Melloni was subsequently led step by step to the assumption of an identity of light with radiant heat. These theories are, however, quite isolated members of the great chain; and so far are we from being able to follow out theoretically the idea of the unity of force, that even now, half a century after Ørsted's discovery, the two electricities are regarded as electrical *fluids*, light as vibrations of *æther*, and heat as motions of the *molecules of bodies*.

> Yet these physical hypotheses are scarcely reconcileable with the idea of the unity of force; and while the latter has had a signal influence on science, this can by no means be said of the former, which have only been useful inasmuch as they furnish a basis for our imagination. Hence it would probably be best to admit that in the present state of science we can form no conception of the physical reason of forces and of their working in the interior of bodies; and therefore (at present, at all events) we must choose another way, free from all physical hypotheses, in order, if possible, to develope theory step by step in such a manner that the further progress of a future time will not nullify the results obtained.
>
> This idea is at the basis, not only of the present investigation, but also of my earlier researches on the theory of light [Lorenz, 1863]; and I am the more moved to adhere to it, that it shows in a remarkable manner how the results which I venture here to develope attach themselves to those I have formerly obtained, and go hand in hand with them. At the same time that I keep the investigation free from all physical hypotheses, I shall endeavour to demonstrate a new member in the chain which connects the various manifestations of the forces; I shall prove that in accordance with the laws for the propagation of electricity under the action of free electricity, and of the electrical currents of the surrounding media, which we can deduce from experiment, periodical electrical currents are possible which in every respect behave like the vibrations of light; from which it indubitably follows that *the vibrations of light are themselves electrical currents.*

In his work, Lorenz starts from the telegraph equation obtained by Kirchhoff [Kirchhoff, 1857]. Lorenz uses for the scalar potential the symbol Ω, for the components of the vector potential (U, V, W), and (u, v, w) for the components of the current, the same symbols used by Weber [Chapter 18, Assis, 2021b]. The equation derives from (a) Weber's force, (b) the recognition of displacement currents in the conductor and (c) Ohm's law for the free moving conducting electricity. Next Lorenz realises that these equations cannot be experimentally distinguished from the delayed potentials he had used in previous works. He therefore adopts them and adopts a value for the delay that corresponds to the velocity experimentally measured by Weber [Chapter 6, Assis, 2021b], indicating that the resulting velocity agrees well with measurements of the velocity of light and that Weber's velocity corresponds to the action of electricity at distance. The final step is to recognise that if the conductor is surrounded by an absolute insulator (no matter what it is and not even if such a thing exists for it is only an idealisation) the currents must show continuity and henceforth be zero at the surface. But such is the condition he had already found in the previous paper. In this sense, light propagates in the vacuum the action (consequences) of the activity in the conductor. The result is the propagation as waves of the potentials that present-day electromagnetism considers given in "Lorenz gauge".

The method used by Lorenz would become later known as "cognitive surpass" and is the method used by children to develop their understanding of the world [Piaget and García, 1989]. It is the method that constructs logic, it is based on synthetic thinking and explicitly disdains fantasies.

In the coming sections we will complete the development of electromagnetism starting from Lorenz ideas and following his epistemic practice.

4.2. On symmetries

In Section 2.2 we introduced the Sensorial World (SW) and the ideal world (IW) as the two poles of a duality that sustains our understanding of Nature. The Sensorial World is what we (as observers) collect through our senses, while the *fact* (as in Piaget) or *reality* (as in Peirce) is the common idea underlying the sensations that we develop through *ideation* (Husserl) or *idealisation* (Galileo) and populate the ideal world. In other words, that what is incorporated in our knowledge as perceived. This starting point has been called "The fundamental antithesis of philosophy" [Ch. I, Whewell, 1858]. We have called the connection between these worlds the *phenomenological map*, i.e., the pair (Π, Γ). Plato emphasised the relation $IW \overset{\Gamma}{\longmapsto} SW$ while Galileo stressed the inverse relation $SW \overset{\Pi}{\longmapsto} IW$.

Although every sensorial input is unique, the projection Π captures that what is present in a phenomenon regardless of circumstances or particularities. The lift Γ reunites reality with the circumstantial or accidental part in observations. Principle 1.1 (NAP) states that when it comes to a description of reality through physical laws, whatever arbitrary elements that are left on IW have to be dealt with in such a way that a change in the choice of arbitrary elements results in an equivalent presentation. Descriptions containing arbitrary elements are connected through a group of transformations relating the different choices of arbitrary elements [Solari and Natiello, 2018].

The introduction of an observer brings about the possibility of attaching to it a Cartesian space for the description of the real and at the same time it introduces the symmetries of the space (the arbitrary element).

Moving directly into electromagnetism, in the terms of its construction we observe that all its fundamental experiments reflect the influence of electromagnetic phenomena associated to a pair of bodies (one of them labelled primary circuit or source or emitter, etc., and the other secondary circuit or receiver or detector). In the same form that space is not a possible subject of experimental detection but spatial relations can be measured, electromagnetic fields can only be detected by their effects on measuring devices, this is: detectors. If the action of a source on a receiver (or detector) can be addressed with controlled degrees of influence from the rest of the universe, in the limit of no influence, the idealised law describing the universe of such relations must depend only on the relative distance and relative motion between source and receiver. Such notions can be found all over the foundational work of Faraday [Faraday, 1839, 1844, 1855] and Maxwell [Maxwell, 1873].

Furthermore, electromagnetic phenomena imply the motion of electricity (whatever electricity is, as Maxwell often said) and then, since what changes the motion of bodies has been called *forces*, we can associate forces with the action of an electromagnetic (EM) body onto another EM body. Actually, this use entails a generalisation of the concept of force, since Newtonian forces change the motional status of macroscopic bodies while microscopic (quantum) objects, such as electrons involved in conduction currents, are not what classical mechanics had in mind when Newton developed its laws. Moreover, if we envisage EM-forces as Lorentz did, by adopting Weber's view of electrical atoms [Lorentz, 1892], such forces must be identically described by observers whose motions relate by Galilean coordinate transformations, and furthermore reciprocal action must be expressed as a symmetry in some privileged systems we call "inertial frames" [Thomson, 1884]. For example, the symmetry inherent to Newton's third law is expressed as the equation $F_{12} + F_{21} = 0$ being invariant in front of Galilean changes of coordinates (where F_{ij} is the force on body j originated in the interaction with body i). Yet, we know at least since [Poincaré, 1900] (see [Solari and Natiello, 2018] as well) that Newton's "action and reaction law" is not compatible with delayed action at distance. Which is then the form this symmetry takes in EM?

When EM theory is moved from its original setting as an *interaction theory* into a *field theory*, some symmetry is broken since there are no longer two EM-bodies in reciprocal action but we are thereafter concerned with only one of them, most frequently the *source*. This presentation of EM may be called the *S-field*. With equivalent arbitrariness we could shift the focus to the receiver and consider an *R-field* description. Both descriptions refer to the same EM phenomena and are therefore related.

When the S-field, the field produced by the source, is perceived by the source itself or by any extended EM-body not moving with respect to the source, we call it *S-by-S-field*. When considering the same S-field as it is perceived by the receiver, we have the *S-by-R-field* description, see Figure 4.2.1. The operation performed on the description of the phenomenon is to identify one body or the other with an extended EM-body in the reference frame of the observer. As both approaches describe the same action, a transformation, possibly dependent on the relative velocity between the EM-bodies, must relate their expressions.

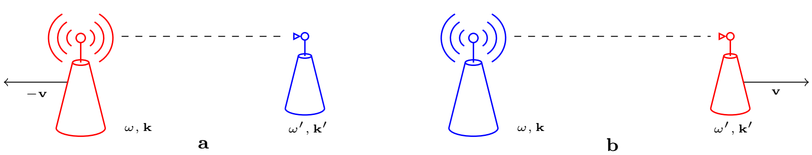

FIGURE 4.2.1. Field of the source (a) as seen by the receiver (S-by-R-field) and (b) as seen by the source (S-by-S-field). Source to the left of each image. In blue: the device at rest with the observer.

For the case of multiple receivers we may want to consider the relation among the different S-by-R$_i$-field descriptions of each receiver. To connect R_1 with R_2 corresponds to the composition of the transformations between each receiver and the source, namely $R_1 \to S$ and (the inverse of) $R_2 \to S$. The composition of transformations yields a transformation between receivers, that will depend on the relative velocities of R_1 and R_2 with respect to the source. However, receiver-receiver transformations relate objects of equivalent character, they are automorphisms and must form a group as well. To complicate things further, to fix our attention in the source is arbitrary, we could have decided to fix our attention on the receiver defining instead the corresponding fields associated to the receiver, namely the *R-field* description with the flavours *R-by-R-field* and *R-by-S-field*, see Figure 4.2.2.

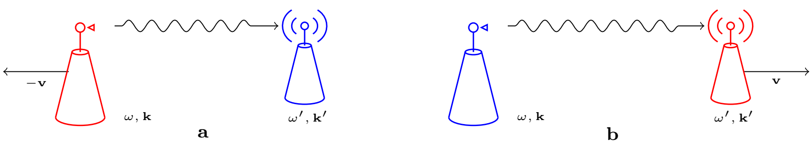

FIGURE 4.2.2. Field of the receiver (a) as seen by the receiver (R-by-R-field) and (b) as seen by the source (R-by-S-field). Source to the left of each image. In blue: the device at rest with the observer.

Hence, in the S-field description a disturbance on the source at a given time will be registered as an alteration on the receiver at a *later* time. In the R-field description, the state of the receiver at a given time will correspond to the state of the source at an *earlier* time. Although the relation source $\longleftrightarrow$ receiver is always causal, it is possible to speak of *advanced* or *retarded* fields in this context.

In the S-field description, considering the disturbance that propagates from the source onto territories at distances that increase with time (where the receiver is) we speak of *retarded* fields. Correspondingly, in the R-field description disturbances propagate from distant territories (where the source lies) onto the receiver. If we insist in interpreting the R-field as the field of a source it would appear as it is propagating backwards in time. Hence, we speak of *advanced* fields.

Returning to relative motion, it must be noticed that even in the case where source and receiver are in constant (also called inertial) relative motion, the transformation between the S-by-S-field and S-by-R-field will not be an inertial transformation (i.e., a Galilean coordinate change). Galilean transformations correspond to descriptive transformations that are not concerned with the observable relative motion of the bodies. On the contrary, the relative motion of source and receiver is a measurable part of the physics involved and not an arbitrariness (it is there independently of the observer). To illustrate this issue consider the following experiment: a source is producing a signal sharply peaked around a given frequency, ω_0 as perceived by a receiver not moving with respect to the source. A set of several, identically built and calibrated receivers are put in motion at various velocities, v_i, with respect to the source, see Figure 4.2.3. How is the signal perceived by each receiver? Which is the perceived characteristic frequency ω_i ? Which is the relation between the signals registered by the various receivers? All these questions belong to the realm of interactions and must have an answer in it.

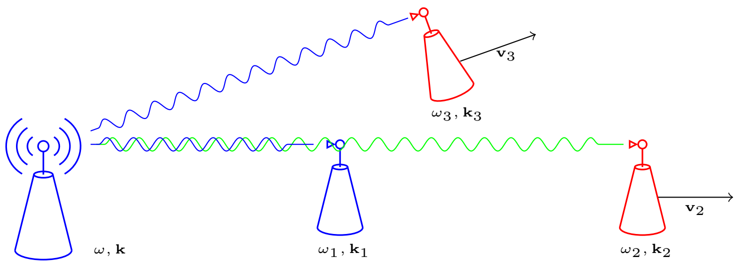

FIGURE 4.2.3. Sources and receivers. Blue receiver at rest relative to source, red receivers in relative motion with respect to the source.

4.3. Relational Electrodynamic Background

4.3.1. Electrodynamics in the spirit of the Göttingen school. We present here a full ether-free derivation of electrodynamics as an interaction theory based in action at distance.

We start with the retarded electromagnetic potentials introduced by Ludwig Lorenz while developing his theory of light as a form of EM interaction corresponding to a transversal wave [Lorenz, 1861a, 1863]:

$$(A, \frac{V}{C})(x,t) = \frac{\mu_0}{4\pi} \int \left(\frac{(j, C\rho)(y, t - \frac{1}{C}|x-y|)}{|x-y|} \right) d^3y, \tag{4.3.1}$$

(cf. the integral in eq. (5) of [Lorenz, 1867]), where μ_0 is the vacuum permeability, ϵ_0 the vacuum permittivity, satisfying the relation $\mu_0\epsilon_0 C^2 = 1$ studied by Weber. C is the velocity associated to the EM waves. In Section (4.1) we discussed the connection of Lorenz' approach with Weber's and Kirchhoff's telegraph equation for the propagation of electromagnetic waves inside a conductor. It is important to indicate that a version of eq. (4.3.1) without delays [p. 122 Assis, 1994] was introduced in [Neumann, 1846] by Franz Neumann and became the starting point for Maxwell's [[542], Maxwell, 1873]. Soon after Neumann's work, Weber repeated the experience (1852) and showed that Neumann's law can be viewed as a result of his (Weber's) force [Chapter 13 Assis, 2021a].

The potentials A, V describe the relation between current-density j or charge-density ρ with their electromagnetic effect. The standard interpretation is that $(j, C\rho)$ are the source (the primary circuit) of the EM action while the potentials are intermediate fields that indicate their (delayed) action over the secondary circuit; this is, $(A, \frac{V}{C})$ are source fields, S-fields.

LEMMA 4.1. *$A(x,t) = \frac{\mu_0}{4\pi}\int_U \left(\frac{j(y, t - \frac{1}{C}|x-y|)}{|x-y|}\right) d^3y \Rightarrow \Box A = -\mu_0 j$, and similarly for $\epsilon_0 \Box V = -\rho$, where $\Box \equiv \Delta - \frac{1}{C^2}\frac{\partial^2}{\partial t^2}$ is the D'Alembert wave operator.*

For a proof, see Appendix 4.A.1. This result goes back to [Lorenz, 1861a, 1867].

Equation

$$\Box A = -\mu_0 j \tag{4.3.2}$$

is an instance of the *wave equation*. Note that the wave property associated to eq. (4.3.1), occurs independently of whether A, V, j, ρ are the electromagnetic potentials, charge and current, or not. Further, Lemma (4.1) is satisfied also by the advanced potentials:

$$(\widetilde{A}, \frac{\widetilde{V}}{C})(x,t) = \frac{\mu_0}{4\pi}\int \left(\frac{(j, \rho C)(y, t + \frac{1}{C}|x-y|)}{|x-y|}\right) d^3y. \tag{4.3.3}$$

that represent a different association for $(j, C\rho)$. They can be interpreted as those corresponding to the secondary circuit and in such a case the potentials express the R-fields that sense an EM perturbation away from the receiver and express its effect later in it; this is, they are *advanced* fields.

The electric and magnetic fields are defined from the potentials as

$$B = \nabla \times A \tag{4.3.4}$$

$$E = -\frac{\partial A}{\partial t} - \nabla V. \tag{4.3.5}$$

Electrodynamics is governed by the least action principle, with the action integral

$$\mathcal{A} = \frac{1}{2}\int dt \int \left(\frac{1}{\mu_0}|B|^2 - \epsilon_0 |E|^2\right) d^3x \tag{4.3.6}$$

as suggested by Lorentz [Lorentz, 1892] inspired in Maxwell's electrokinetic energy and potential energy [[630,631] and [634,635], Maxwell, 1873][49].

The main result of electrodynamics can be stated as follows.

THEOREM 4.1. *Let (A, V) be the known values of the electromagnetic potentials in a piece of matter supported on a region of space with characteristic function*

[49]It is important to understand that Lorentz made a fusion of Maxwell's theory and delayed action at distance although he continued to call it Maxwell's theory. Lorentz says:

> In general, the assumptions that I introduce represent in a certain sense a return to the earlier theories of electricity. The core of Maxwell's views is therefore not lost, but it cannot be denied that with the adoption of ions we are not far away from the electric particles, which were used earlier.[Lorentz, 1895]

χ. *Then, assuming that all of* $|B|^2, |E|^2, A, j, V, \rho$ *decrease faster than* $\frac{1}{r^2}$ *at infinity, Hamilton's principle of least action [Ch 3, 13 A p. 59, Arnold, 1989],* $\delta\mathcal{A} = 0$, *subject to the constraints given by* (A, V) *implies the relations*

$$\epsilon_0 \nabla \cdot E = \rho \tag{4.3.7}$$

$$\mu_0 j + \frac{1}{C^2}\frac{\partial E}{\partial t} = \nabla \times B \tag{4.3.8}$$

$$\nabla \cdot j + \frac{\partial \rho}{\partial t} = 0. \tag{4.3.9}$$

COROLLARY 4.1. *In the special case where the relation* $\nabla \cdot A + \frac{1}{C^2}\frac{\partial V}{\partial t} = 0$ *(the "Lorenz gauge") is satisfied, the manifestation of the potentials outside matter obeys the wave equation, eq.* (4.3.2).

We develop the proof in Appendix 4.A.2.

Theorem (4.1) deserves to be named *Lorenz-Lorentz theorem* since in Lorenz' conception light was associated to the EM activity inside matter [Lorenz, 1867] and Lorentz proposed the expression for the action based on Maxwell's energy considerations.

The least action principle for the action integral of eq. (4.3.6) completes the derivation of the four Maxwell equations, namely eqs. (4.3.4, 4.3.5, 4.3.7, 4.3.8) as well as the continuity equation, eq. (4.3.9), proving also gauge invariance and the possibility of electromagnetic waves. No reference whatsoever to the ether has been used and the phenomenological background is rooted in charges and currents as intuited by the Göttingen school.

Finally, the observable effects of electromagnetic interactions over the electrified particles that constitutes matter can be traced back to (interpreted as the result of) the action of the *Lorentz force* [Lorentz, 1892, Natiello and Solari, 2021]. This force can also be derived from the least action principle, as we presently discuss in Subsection 4.3.5.

The action integral can be equivalently stated using Lorenz potentials:

LEMMA 4.2. *Up to an overall function of time and the divergence of a function vanishing sufficiently fast at infinity (which do not contribute to the variation of the action integral), the electromagnetic action satisfies*

$$\begin{aligned} \mathcal{A} &= \frac{1}{2}\int dt \int \left(\frac{1}{\mu_0}|B|^2 - \epsilon_0 |E|^2\right) d^3x \\ &= \frac{1}{2}\int dt \int (A \cdot j - \rho V)\, d^3x \end{aligned} \tag{4.3.10}$$

PROOF. The proof requires standard vector calculus operations on the integrand, namely

$$\begin{aligned}
A\cdot j-\rho V &= A\cdot\left(\frac{1}{\mu_0}\nabla\times B-\epsilon_0\frac{\partial E}{\partial t}\right)-\epsilon_0 V\nabla\cdot E\\
&= \frac{1}{\mu_0}\left(|B|^2-\nabla\cdot(A\times B)\right)-\epsilon_0 A\cdot\frac{\partial E}{\partial t}-\epsilon_0\nabla\cdot(VE)+\epsilon_0 E\cdot\nabla V\\
&= \frac{1}{\mu_0}\left(|B|^2-\nabla\cdot(A\times B)\right)-\epsilon_0 A\cdot\frac{\partial E}{\partial t}-\epsilon_0\nabla\cdot(VE)-\epsilon_0 E\cdot\left(E+\frac{\partial A}{\partial t}\right)\\
&= \frac{1}{\mu_0}|B|^2-\epsilon_0|E|^2-\nabla\cdot\left(\frac{1}{\mu_0}A\times B+\epsilon_0 VE\right)-\epsilon_0\frac{\partial}{\partial t}(A\cdot E)
\end{aligned}$$

□

Moreover, the dynamics given by the least action principle can be shown to be independent of the so-called *gauge transformations* $A' = A + \nabla\Lambda$ and $V' = V - \dfrac{\partial\Lambda}{\partial t}$ (for an appropriate function Λ).

LEMMA 4.3. *The result of Lemma 4.2 is independent of the choice of gauge.*

PROOF. Modifying the potentials with a sufficiently smooth function $\Lambda(x,t)$ vanishing adequately at infinity and such that $A' = A + \nabla\Lambda$ and $V' = V - \dfrac{\partial\Lambda}{\partial t}$, we obtain

$$\begin{aligned}
A'\cdot j-\rho V' &= A\cdot j+(\nabla\Lambda\cdot j)-\rho(V-\frac{\partial\Lambda}{\partial t})\\
&= A\cdot j-\rho V+\left(\nabla\Lambda\cdot j+\rho\frac{\partial\Lambda}{\partial t}\right)\\
&= A\cdot j-\rho V+\left(\nabla\cdot(\Lambda j)-\Lambda\nabla\cdot j+\rho\frac{\partial\Lambda}{\partial t}\right)\\
&= A\cdot j-\rho V+\nabla\cdot(\Lambda j)+\frac{\partial}{\partial t}(\rho\Lambda)
\end{aligned}$$

Hence, the action integral computed through both expressions differ only in an overall function of time and the volume integral of a divergence and Lemma 4.2 applies. □

4.3.1.1. *Maxwell's energy revisited.* Maxwell's energy is introduced through a process in which matter acquires its electromagnetic state [Maxwell, 1873]. The electrostatic energy is obtained in [630-631] bringing arbitrarily small charges from infinity. This process will be criticised in the context of microscopic entities, in Chapter 5 Next, the magnetostatic energy is obtained in similar form in [632-633], based upon magnetostatic results previously obtained in [389]. Maxwell proceeds to add an electrokinetic energy due to the currents [634-635]. In the even numbered articles he presents the physical idea and in the odd numbered articles he transforms the expression using integration by parts.

Thus, the energy required to create a given electromagnetic state (a distribution of charges and currents) can be regarded as the time-integral of the power, first bringing charges from a condition of zero energy (from "infinity") working against $-\nabla V$ and bringing also the current distribution, now working against the electromagnetic momentum A:

$$\mathcal{P} = \left(\nabla V + \frac{\partial A}{\partial t}\right) \cdot j = -E \cdot j$$

the time-integral from a situation in which $\mathcal{E}(0) = 0$, leads to

$$\mathcal{E} = -\int_0^t dt \int d^3x \,(E \cdot j) = \int_0^t dt \int d^3x \left(\left(\nabla V + \frac{\partial A}{\partial t}\right) \cdot j\right). \tag{4.3.11}$$

Assuming that all of $|B|^2, |E|^2, A \cdot j, V\rho$ decrease faster than $\frac{1}{r^2}$ at infinity (a hypothesis needed for most manipulations performed by Maxwell and Lorentz) the following Lemma holds:

Lemma 4.4. *Eq. (4.3.11) is equivalent to*

$$\mathcal{E} = \frac{1}{2} \int d^3x \left(\frac{1}{\mu_0}|B|^2 + \epsilon_0 |E|^2\right) \tag{4.3.12}$$

(the total electromagnetic energy), up to the volume integral of the gradient of a function that vanishes at infinity.

From the electrostatic and magnetostatic situations it is clear that the individual terms correspond to the electrokinetic energy $T = \frac{1}{2}\int d^3x \left(\frac{1}{\mu_0}|B|^2\right)$ and the potential energy $U = \frac{1}{2}\int d^3x \left(\epsilon_0 |E|^2\right)$, thus suggesting that the electromagnetic Lagrangian reads $\mathcal{L} = T - U$ and the action $\mathcal{A} = \int dt\, \mathcal{L}$.

4.3.2. Symmetries of the potentials and of the action integral. We recast the potentials of eq. (4.3.1) as the convolution of charge and currents with the *Lorenz kernel* hereby defined:

$$K(x - y, s - r) = \frac{1}{|y - x|}\delta(s - r - \frac{1}{C}|y - x|),$$

namely

$$(A, \frac{V}{C})(x, s) = \frac{\mu_0}{4\pi} \int \left[\int_{-\infty}^{t} K(x - y, s - r)(j, C\rho)(y, r)\right] d^3y\, dr \tag{4.3.13}$$

The expression gives the values of the potentials generated by a source (j, ρ) at a point and time (x, s) where a probe or target can be placed. The fundamental symmetry between potentials and wave operators is expressed in the following

Lemma 4.5. *The action of the kernel $K(x - y, s - r)$ and the differential operator $\Box$ are reciprocally inverse of each other.*

Proof. We discuss the proof using A to fix ideas, and write eq. (4.3.13) in shorthand as $A = \dfrac{\mu_0}{4\pi} K * j$ (where the star stands for convolution). Composition with $\Box$ gives:

$$\begin{aligned} \Box A &= \frac{\mu_0}{4\pi}\Box K * j = -\mu_0 j \\ K * \Box A &= -\mu_0 K * j = -4\pi A. \end{aligned}$$

Hence, in their respective domain of definition $\Box K = -4\pi Id$ (convolution identity) and $K * \Box = -4\pi Id$ (operator identity). $\Box$

Since the action plays a fundamental role in this relational presentation we should devote some lines to consider its symmetries as well.

We first write the action in terms of definite integrals and the kernel $K(x-y, s-r)$

$$\mathcal{A} = \frac{1}{2}\frac{\mu_0}{4\pi}\int_{t_0}^{t} ds \int_{t_0}^{t} dr \iint K(x-y, s-r)\left(j_1 \cdot j_2 - C^2 \rho_1 \rho_2\right) d^3x\, d^3y. \quad (4.3.14)$$

where indices 1(2) label a source (respectively a target) in interaction. The form of the action in eq. (4.3.14) is almost symmetric in terms of exchanging source (primary) and target (secondary) circuits. Interchanging primary and secondary circuit, and $(x, s) \longleftrightarrow (y, r)$ the kernel changes into

$$K(x-y, s-r) = \frac{1}{|y-x|}\delta(s - r + \frac{1}{C}|y-x|) \quad (4.3.15)$$

Thus, the action considered is always the action of the primary circuit over the secondary circuit which can be written in two forms. In one of them, the S-field (the standard form), EM changes are propagated with delay by the potentials (and their derivatives, the EM-fields) at distances away from the source. The symmetry related form, the R-field, associates an advanced field with the target receiver. In this form, the field can be seen as a sensor that will carry disturbances to the receiver that will display changes at a later time.

The symmetry of the action has the immediate consequence that all lemmas and theorems of Section (4.3.1) have an equivalent form under this symmetry operation. In particular, there is Lorentz force where the S-fields, R-currents and R-charges are exchanged by R-fields, S-currents and S-charges. This relation is what corresponds to the action and reaction law for actions that propagate instantaneously, since in the limit $C \to \infty$ the S-field and the R-field of a given body/device coincide.

4.3.3. Detection/perception in relative motion . Let us consider the potentials A, V originated in a source with current-charge $J = (j, C\rho)$ measured at rest relative to the source (with coordinate y). We consider further a detector extending over a variable x with reference to a distinguished point in it. In the case of source and detector at relative rest, we write

$$(A, \frac{V}{C})(x,t) = \frac{\mu_0}{4\pi}\int d^3y \int ds \left(\frac{\delta\left(t - s - \frac{1}{C}|x-y|\right)}{|x-y|}\right) J(y,s) \quad (4.3.16)$$

$$= \frac{\mu_0}{4\pi}\int d^3z \left(\frac{J(x-z, t-\frac{1}{C}|z|)}{|z|}\right) \quad (4.3.17)$$

These equations are formulated under the following premises: Coordinates y and x are described from the same spatial reference system S, whatever it is, and hence at a given time t, $x - y$ and in particular $|x - y|$ are objective invariant quantities. Moreover, since source and detector are in relative rest, these quantities are independent of t. In the present conception of electromagnetism there is another objective invariant quantity of relevance, namely the electromagnetic delay $\Delta_0 = t - s$. The index 0 highlights the situation of relative rest between source and detector. It is the state of point y on the source at the previous time s, where $C(t-s) = |x-y|$ what connects with point x on the detector at time t. Finally, the second row displays the change of variables $z = x - y$.

Given the potentials in the form of eq. (4.3.16) and the action (eq. (4.3.14)) the action can be written in terms of the R-potentials as:

$$\mathcal{A} = \frac{\mu_0}{4\pi} \int ds \left[\int \left((\widetilde{A}_2(y,s) \cdot j_1(y,s) - \widetilde{V}_2 \rho_1(y,s)) \right) d^3y \right] \tag{4.3.18}$$

which is the symmetry companion of eq. (4.3.10). Although we will restrict our discussion to the S-fields, the symmetry of the action indicates that there is a symmetry-related discussion with the R-fields.

In order to address detection in relative motion we advance the following

Conjecture 4.1. *A detector recording solely electromagnetic information (e.g. an electromagnetic wave) cannot determine its relative velocity with respect to the source (assumed constant).*

Consequently, let us postulate that a detector in relative motion with velocity v with respect to the source perceives an EM wave as if it was originated in some (possibly different) current-charge *at relative rest*. We would like to show something like:

$$(A, \frac{V}{C})_v(x,t) \quad = \quad \frac{\mu_0}{4\pi} \int d^3y \left(\frac{1}{|x-y|} \right) J_v(y, t-\Delta) \tag{4.3.19}$$

with $\Delta = \frac{1}{C}|x-y|$, for some effective current-charge J_v to be specified.

In this new situation we still have one reference frame S to describe both source and detector. Again, $z = x - y$ is an objective quantity, only that now two differences arise: (a) $x - y$ depends on t because of the relative motion and (b) the electromagnetic delay may be modified in order to take into account the relative motion. Throughout this discussion, t is the (present) time when the electromagnetic interaction is detected, $(x-y)$ indicates the relative position of (points of) detector and source at time t, $\Delta_v = (t-s)_v$ is the electromagnetic delay, $(x-y)_v$ is the corresponding relative position at time s when the electrical disturbance in the source took place, and the index $v \in \mathbb{R}^3$ indicates a situation of relative motion between source and detector. The index v will be some function of the relative velocity u between source and detector to be determined in what follows. Moreover, $(x-y)_v$ and Δ_v are objective and invariant quantities, independent of the choice of reference frame.

We intend to find the correspondence between disturbances in the primary circuit and actions on the secondary system. We begin by considering an infinitesimal velocity δv, with $\frac{d\delta v}{dt} = 0$. In this case we have

Definition 4.1. **(Differential delayed interaction condition)** In the presence of relative motion with infinitesimal velocity δv, a disturbance originated at point y and time $t-\Delta_{\delta v}$ produces an electromagnetic action at (x,t), where

$$C\Delta_{\delta v} \quad = \quad |x-y-\Delta_{\delta v}\delta v|.$$

For $\delta v=0$ the condition reduces to $C\Delta_0=|x-y|$, corresponding to Lorenz' potentials, eqs. (4.3.1) and (4.3.13)[50]. Note that C enters in both expressions since we postulate that the detector in relative motion registers an electromagnetic signal *as if the source were at relative rest.* This definition leads to the following

LEMMA 4.6. *Let $(x-y)_v$ be the separation of source and detector at time s when the electrical disturbance at the source took place in a situation of relative motion labelled by $v\in\mathbb{R}^3$ and Δ_v the corresponding electromagnetic delay, while $(x-y)_0, \Delta_0$ are the corresponding quantities for source and detector at relative rest. Then, for each v the* ***delayed interaction condition*** *satisfies*

$$\begin{pmatrix} (x-y) \\ C\Delta \end{pmatrix}_v = \exp\left(-\begin{pmatrix} \mathbf{0} & \frac{v}{C} \\ \frac{v}{C}^T & 0 \end{pmatrix}\right)\begin{pmatrix} (x-y) \\ C\Delta \end{pmatrix}_0$$

PROOF. To lowest order in δv the difference in Δ's is:

$$\begin{aligned} C\left(\Delta_{\delta v}-\Delta_0\right) &= \sqrt{|x-y|^2-2\,(x-y)\cdot\delta v\Delta_{\delta v}+|\delta v|^2\Delta_{\delta v}^2}-|x-y| \\ &= -\frac{(x-y)}{|x-y|}\cdot\delta v\Delta_0+O(\delta v^2)=-\,(x-y)\cdot\frac{\delta v}{C}+O(\delta v^2) \end{aligned}$$

In this limiting case the condition reads

$$\begin{aligned} \begin{pmatrix} (x-y)_{\delta v} \\ C\Delta_{\delta v} \end{pmatrix} &= \begin{pmatrix} (x-y)-\delta v\Delta_0 \\ C\Delta_0-(x-y)\cdot\frac{\delta v}{C} \end{pmatrix} \qquad (4.3.20) \\ &= \left[\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}-\begin{pmatrix} 0 & \frac{\delta v}{C} \\ (\frac{\delta v}{C})^T & 0 \end{pmatrix}\right]\begin{pmatrix} (x-y) \\ C\Delta_0 \end{pmatrix}. \end{aligned}$$

In other words, there exists an infinitesimal transformation on $\mathbb{R}^{3+1}$ connecting the condition for $v=0$ with that for δv. By the Trotter product formula (or just by plain exponentiation) we obtain Lie's result for finite v as a repeated composition of infinitesimal shifts,

$$\begin{aligned} TL(-v) &\equiv \exp\left(-\begin{pmatrix} \mathbf{0} & \frac{v}{C} \\ \frac{v}{C}^T & 0 \end{pmatrix}\right) \\ &= \lim_{n\to\infty}\left[\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}-\frac{1}{n}\begin{pmatrix} \mathbf{0} & \frac{v}{C} \\ \frac{v}{C}^T & 0 \end{pmatrix}\right]^n \qquad (4.3.21) \end{aligned}$$

thus proving the statement. □

[50]Letting $s=t-\Delta_{\delta V}$ we may read the definition as a consequence of: $(x-y)(s)=(x-y)(t)-(t-s)\delta v$.

REMARK 4.1. Explicit formulae for the Lorentz transformations are shown in the Appendix 4.B. The more familiar form $L(u)$ of the transformation is displayed in

$$\left(\begin{array}{c} z_u \\ C\Delta_u \end{array}\right) = L(u)\left(\begin{array}{c} z \\ C\Delta_0 \end{array}\right) = \left(\begin{array}{c} z + (\gamma - 1)\hat{u}\,(\hat{u}\cdot z) + \gamma \dfrac{u}{C} C\Delta_0 \\ \gamma \left(C\Delta_0 + \dfrac{u \cdot z}{C}\right) \end{array}\right) \quad (4.3.22)$$

where $u = C\hat{v}\tanh\left|\frac{v}{C}\right|$, $\gamma \equiv \gamma(u) = \left(1 - \frac{u^2}{C^2}\right)^{-1/2}$ and we use the shorthand $x - y = z$. There is a 1-to-1 correspondence in Lemma 4.6, between the two presentations of the Lorentz transformations, namely $TL(-v) \equiv L(-u)$. Hence, we will use only u in the sequel. u is interpreted as the relative velocity between source and detector.

The basis for the interpretation of u as the relative velocity is as follows. Consider the vector space $\mathbb{R}^{3+1} \equiv \mathbb{R}^3 \times \mathbb{R}$ associated to relative positions and relative time. A Lorentz transformation (LT), eq. (4.3.22), as well as a Galilean transformation GT,

$$\left(\begin{array}{c} Z' \\ T' \end{array}\right) = \left(\begin{array}{cc} \mathbf{1} & u \\ 0 & 1 \end{array}\right)\left(\begin{array}{c} Z \\ T \end{array}\right)$$

can be regarded as endomorphisms of $\mathbb{R}^{3+1}$ mapping a situation at relative rest onto a situation of relative motion. While the velocity u in the GT has a mechanical origin, the parameter u in LT is an abstract parameter used to classify transformations and a point of contact with the underlying physical problem is required to furnish a physical interpretation to the LT's. Considering lines on $\mathbb{R}^{3+1}$ associated to a fixed relative position, Z and different time-intervals, we obtain for the Galilean transformation the (physical) relative velocity $u = \frac{Z'(T_1)-Z'(T_0)}{T'(T_1)-T'(T_0)}$ while in the case of the Lorentz transformation we obtain

$$\frac{z'(\tau_1) - z'(\tau_0)}{\tau'(\tau_1) - \tau'(\tau_0)} = \frac{\gamma u(\tau_1 - \tau_0)}{\gamma(\tau_1 - \tau_0)} = u.$$

Note that while the GT preserves times and as such can be viewed as a transformation in relational space only, the LT preserves $|z|^2 - (C\tau)^2$ and, as a particular case, the condition of being in electromagnetic contact, $|z|^2 - (C\tau)^2 = 0$. We may associate the same relational velocity to both GT and LT.

Eq. (4.3.21) displays the action of a Lorentz boost [Gilmore, 1974] in the Lie algebra (rhs) and group (lhs). The generators of the Lorentz boosts plus the generators of the rotations constitute the basis of the Lie algebra which exponentiated gives the Poincare-Lorentz group. While the spatial rotations form a subgroup of the Poincaré-Lorentz group, the Lorentz boosts do not. Any element of the Poincaré-Lorentz group can be written as a product: $P = L(u)R(\Omega)$ as well as $P = R(\Omega)L(u')$ being Ω a 3d-rotation and $u' = R(\Omega)u$. These forms are known as left and right coset decompositions of the group [Hamermesh, 1962, Gilmore, 1974].

REMARK 4.2. By construction of the $LT's$, there is an upper limit for having electromagnetic contact amenable to be related with situations at relative rest.

While there is no mechanical limit to relative velocity, the present theory describes electromagnetic interactions only for $|u| < C$.

REMARK 4.3. Eqs. (4.3.20) and (4.3.22) for the detector and source points, x, y which are in electromagnetic interaction at time t, display their relative position $(x-y)_u$ at the time $t - \Delta_u$ when the disturbance in the source took place. The ratio $\frac{|(x-y)_u|}{\Delta_u} = C$ is always satisfied by construction.

Next, we note that the propagation kernel can be more properly written as

$$K = \begin{cases} 0, & (t-s) < 0 \\ \dfrac{\delta(t-s-\frac{1}{C}|x-y|)}{|x-y|}, & (t-s) \geq 0. \end{cases}$$

Hence, we have the following

LEMMA 4.7. ***(Symmetric form of the propagation kernel)*** *Lorenz propagation kernel can be rewritten as*

$$K = \begin{cases} 0, & (t-s) < 0 \\ \dfrac{2}{C}\delta((t-s)^2 - \dfrac{1}{C^2}|x-y|^2), & (t-s) \geq 0. \end{cases} \tag{4.3.23}$$

PROOF. In the distribution sense $K = \dfrac{2|x-y|}{C(t-s)+|x-y|}K$. By another distributional property, for any $g(s)$ such that $g(s_0) \neq 0$ it holds that $\dfrac{\delta(s-s_0)}{|g(s)|} = \delta(g(s)(s-s_0))$. In this case, $g(s) = t - s + \dfrac{1}{C}|x-y|$. Hence, we obtain the symmetric kernel expression of eq. (4.3.23). □

THEOREM 4.2. *The Lorenz propagation kernel $K(x,t;y,s)$ has the following properties in relation to Lorentz transformations*

$$\begin{aligned} K(L_u(x,t); L_u(y,s)) &= K(x,t;y,s) \\ K(L_u(x,t); y,s) &= K(x,t; L_{-u}(y,s)) \\ \int d^3y\, ds\, [K(L_u(x,t); y,s)J(y,s)] &= \int d^3y\, ds\, [K(x,t;y,s)J(L_u(y,s))] \end{aligned}$$

The last equation reads: the transformation of the potentials are the potentials associated to the transformations of the currents. We say then that the linear operator associated with K commutes with the Lorentz transformation.

PROOF. It is straightforward to verify that the argument of the δ-distribution in eq. (4.3.23) is invariant upon Lorentz transformations, namely that if $((x-y)_u, C(t-s)_u)$ satisfy eq. (4.3.22), then $(t-s)^2 - \dfrac{1}{C^2}|x-y|^2 = (t-s)_u^2 -$

$\frac{1}{C^2}|(x-y)_u|^2$ and also $(t-s) \geq 0 \Longleftrightarrow (t-s)_u \geq 0$. Thus,

$$K = \begin{cases} 0, & (t-s)_u < 0 \\ \frac{2}{C}\delta((t-s)_u^2 - \frac{1}{C^2}|\,(x-y)_u\,|^2), & (t-s)_u \geq 0. \end{cases}$$

is independent of u. Using the first property it follows that $K(L_u(x,t);y,s) = K(L_u(x,t;L_uL_{-u}(y,s)) = K(x,t;L_{-u}(y,s))$. The commutation relation is the result of integrating the kernel to produce a linear operator and changing integration variables $((y,s) \mapsto L_u(y',s')$. □

REMARK 4.4. The points that are in electromagnetic connection are characterised by $(C(t-s))^2 - |x-y|^2 = 0$. Calling $\tau_u \equiv (t-s_0)_u$ and $\chi_u \equiv (x-y)_u$, the interaction kernel is the convolution kernel of $\delta(\tau_u^2 - (\chi_u/C)^2)$ which can be split in two contributions, one for $\tau_u \geq 0$ and another for $\tau_u \leq 0$. But, if $(0,0)$ is influencing (τ_0, χ_0) for $\tau_0 \geq 0$, it results that $\tau_u > 0$ (using that $|u \cdot x/C^2| = \frac{|u\cdot\chi|}{|\chi||u|}\frac{|u||\chi|}{C^2} \leq \frac{|u||\chi|}{C^2}$) hence the splitting is really in terms of *influencing*, $\tau_u \geq 0$, vs. *being influenced,* $\tau_u \leq 0$. This separates the sets in a form invariant with respect to u.

4.3.3.1. *Perceived fields and inferred currents-charges.* If we further examine eq. (4.3.19), we note that it represents a convolution product with convolution kernel $\kappa(z,r)$, with $K(x,t;y,s) = \kappa(x-y,t-s)$ and that

$$(A, \frac{V}{C})_u = \frac{4\pi}{\mu_0}\kappa * J_u = \frac{4\pi}{\mu_0}J_u * \kappa$$

where the convolution is in time and space.

According to eq. (4.3.20), the arguments in the current are $(x-y, t-s)$, for $u=0$. For $u \neq 0$ the points that are in electromagnetic relation according to Lemma 4.6 are $((x-y)_u, (t-s)_u)$, thus in $J_u * \kappa$, we propose

CONJECTURE 4.2. *The arguments of the effective current are $((x-y)_u, (t-s)_u)$, i.e., $J_u = J(L(u)(x-y,t-s))$, where J is the current-charge measured by the source.*

At this point we must notice that there are three forms in which current-charge can be transformed to produce a new pair satisfying the continuity equation. Two of them are Galilean:

$$\begin{aligned}
(j, C\rho)(x,t) &= (j - v\rho, C\rho)(x+vt, t) && (4.3.24)\\
(j, C\rho)(x,t) &= (j, C\rho - \frac{v}{C}\cdot j)(x, t + \frac{v\cdot x}{C^2}) && (4.3.25)\\
(j, C\rho)_u(x,t) &= L(-u)(j, C\rho)(L(u)(x,t)) && (4.3.26)
\end{aligned}$$

Note that in the third one, the leftmost L acts on the current-charge $4D$-vector while the rightmost acts on the space-time coordinates.

If the form eq. (4.3.24) is adopted, a theorem due to Maxwell [[602] Maxwell, 1873] shows that from the point of view of the receiver the transformation (4.3.25) must be applied to preserve the mechanical force but in such case the perceived potentials/fields are not waves[51]. The empirical evidence has judged this view as not correct.

We then propose to adopt eq. (4.3.26) as a definition of the inferred current. We insist at this point that the symmetry is not an a posteriori observation of the formulae, but rather an a priori demand of constructive reason as explained in NAP [Solari and Natiello, 2018]. The transformation of current-charge presents itself as a demand of reason to be later confronted with empirical results. That a charge density in motion can be perceived as a current is a belief firmly adopted since Weber's electrodynamic studies [Weber, 1846] and we are habituated to accept it, while that a neutral current in motion will be perceived as charge is not rooted in our beliefs in the same way, despite the fact that Maxwell's theorem already opened for that possibility.

Remark 4.5. The symmetric form of K is especially appealing when consider the backwards propagation kernel, as in the equation pairs (4.3.1)–(4.3.3) and (4.3.13)–(4.3.15). The backward propagation kernel is the result of inverting the time inequalities in 4.3.23.

Remark 4.6. We may be tempted to ask: which is the meaning of a successive application of Lorentz transformations to a current? The only meaning we find apt is that if $J_u = L_{-u}J(L_u(x,t))$, then $J = L_u J_u(L_{-u}(x,t))$ (since Lorentz transformations have as inverse the transformation based on minus the velocity) and correspondingly $J_{u'} = L_{-u'}L_u J_u(L_{u'}L_{-u}(x,t))$. Since $L_{u'}L_{-u}$ is a general element of the Poincaré-Lorentz group, $L_{u'}L_{-u} = L_{u'\ominus u}R(u',u)$ with $u' \ominus u$ the coset addition of velocities, also known as Einstein's addition [Gilmore, 1974] and $R(u',u)$ a Wigner rotation[52] . Thus, the Poincaré-Lorentz group allows to convert between inferred currents or fields associated to different detectors in relative motion with respect to the same source. Notice that the relative velocity between both receptors is $u' - u$ but the correspondence of electromagnetic perceptions is not $L(u' - u)$ which might even not exist.

Next, we explore the consequences of this proposal. Let us define operators acting on scalar or vector (when appropriate) functions, J, of (x,t) as

$$\begin{aligned} \widehat{K}\,[J]\,(x,t) &\equiv \iint d^3z\, d\Delta\, K(z,\Delta)J(x-z,t-\Delta) \qquad (4.3.27)\\ \widehat{L}_u\,[J] &\equiv J(L(u)(x,t)) \\ (\widehat{A}\circ\widehat{B})[J] &\equiv \widehat{A}\left[\widehat{B}[J]\right] \end{aligned}$$

The first line defines the action of the propagating kernel as a convolution, the second the action of a Lorentz transformation on the coordinates (recall that $u = Cv\tanh\left|\frac{v}{C}\right|$) while the third relation establishes notation.

[51] In Maxwell-Hertz terms, this form corresponds to an ether dragged by the source.

[52] Actually, Wigner was not the first to study the group structure associated to Lorentz transformations or to mention the rotation. At least Silberstein [p. 167, Silberstein, 1914] in the published notes of his 1912-1913 course on relativity at the University College, London, preceded Wigner, who acknowledged this precedence.

Lemma 4.8. *According to the previous discussion, the perceived potentials read*

$$(A, \frac{V}{C})_u = \widehat{K}\left[J_u\right] \tag{4.3.28}$$

In addition, we have the following identity

$$\widehat{K}\left[J_u\right] \quad = \quad L(-u)\widehat{L}_u\left[\widehat{K}\left[J\right]\right]$$

Proof. Note that $L(-u)$ acts on the current-charge $J = (j, C\rho)$, while $\widehat{L}_u$ acts on the spatial/temporal arguments x, Ct. Eq. (4.3.28) is just eq. (4.3.19) rewritten through eq. (4.3.27). Recalling from eq. (4.3.26) that $J_u = L(-u)\widehat{L}_u\left[J\right]$ and from Theorem 4.2 that $\widehat{L}_u \circ \widehat{K} = \widehat{K} \circ \widehat{L}_u$ and finally that the matrix $L(-u)$ commutes with the scalar operator K we obtain the result. □

Corollary 4.2. *By Theorem 4.2 and Lemma 4.8, the action integral is invariant in front of perception transformations, i.e.,*

$$\mathcal{A} \quad = \quad \frac{1}{2}\int dt\, d^3x\ \left((A, V/C)_1 \odot (j, C\rho)_2\right) = \frac{1}{2}\int dt\, d^3x'\ \left((A', V'/C)_1 \odot (j', C\rho')_2\right)$$

where

$$\int (A, V/C) \odot (j, C\rho) = A \cdot j - V\rho = \int \begin{pmatrix} A & V/C \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j \\ C\rho \end{pmatrix}$$

Proof. By the properties of $\widehat{K}$ and the Lorentz transformations, we have that

$$L(-u)\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 1 \end{pmatrix} L(-u) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned}
\int d^3x'dt'\ \left(A'_1 \cdot j'_2 - V'_1\rho'_2\right) &= \int d^3x'dt'\ \begin{pmatrix} j & C\rho \end{pmatrix}_1 L(-u)\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 1 \end{pmatrix} L(-u)\begin{pmatrix} j \\ C\rho \end{pmatrix}_2 \\
&\quad K(\widehat{L}_u(x', t); \widehat{L}_u(y', s)) \\
&= \int d^3x dt\ \begin{pmatrix} j & C\rho \end{pmatrix}_1 \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} j \\ C\rho \end{pmatrix}_2 K(x, t; y, s) \\
&= \int d^3x\, dt\ (A_1 \cdot j_2 - V_1\rho_2)\,.
\end{aligned}$$

□

4.3.3.2. *The Doppler effect.* The perception of wave frequencies in the case the waves are produced by a source in relative motion with respect to the receptor is known as *Doppler effect.* The EM Doppler effect plays a fundamental role in physics [Dingle, 1960b, Mandelberg and Witten, 1962, Kaivola et al., 1985]. The goal of this section is to show that the present theory provides an explanation for the experimental observations of the Doppler effect. To begin with, all Doppler experiments consist in comparing the waves perceived by a detector at rest with respect to the source against the perception of a detector moving at constant velocity (within acceptable experimental precision) relatively to the source.

In practice, the task is to obtain the Fourier transform of eq. (4.3.19). We will keep track of this process conceptually, and hence it is better to use the operator notation from Lemma 4.8. The Fourier transform of a function will be:

$$\mathcal{F}_{k,w}\left[\phi\right] = \frac{1}{(2\pi)^2}\iint d^3x\, dt\, \exp\left(-i(k\cdot x - wt)\right)\phi(x,t)$$

and is a function of (k, w), where we have made an arbitrary choice in the election of the sign preceding wt (that does not influence the conclusion). We will use the following known results:

$$\begin{aligned}\mathcal{F}_{k,w}\left[\widehat{L}_u\phi\right] &= \mathcal{F}_{k',w'}\left[\phi\right] \text{ , with } (k', \frac{w'}{C}) = L(-u)(k, \frac{w}{C})\\ \mathcal{F}_{k,w}\left[\widehat{K}\phi\right] &= \frac{1}{w^2 - C^2k^2}\mathcal{F}_{k,w}\left[\phi\right]\end{aligned}$$

The first result is the immediate consequence of $L(u)$ being symmetric, while the second one can be obtained in various ways including direct integration. Applying these results to eq. (4.3.28) we obtain

$$\begin{aligned}\mathcal{F}_{k,w}\left[\widehat{K}[J_u]\right] &= \mathcal{F}_{k,w}\left[\widehat{K}\left[L(-u)\widehat{L}_u\left[J\right]\right]\right]\\ &= L(-u)\mathcal{F}_{k,w}\left[\widehat{L}_u\left[\widehat{K}\left[J\right]\right]\right]\\ &= L(-u)\mathcal{F}_{k',w'}\left[\widehat{K}\left[J\right]\right]\\ &= L(-u)\frac{1}{w'^2 - C^2k'^2}\mathcal{F}_{k',w'}\left[J\right]\\ &= L(-u)\frac{1}{w^2 - C^2k^2}\mathcal{F}_{k',w'}\left[J\right]\end{aligned}$$

where $(k', \frac{w'}{C}) = L(-u)(k, \frac{w}{C})$. Thus, in terms of wave frequencies, the Fourier spectrum will have a peak at $w' = \gamma(u)(w - k\cdot u)$ associated with a source of frequency w. The primed quantities describe the characteristics of the wave as perceived by the detector while the unprimed refer to the source. When $k\cdot u = |k||u|$ the relative distance between source and detector increases, $w' < w$, and correspondingly the wavelength shifts towards higher values (red shift).

Hence, we have proved the following

Theorem 4.3. ***(Doppler effect)*** *A detector (observer) in relative motion with velocity u with respect to an electromagnetic source emitting current-charge waves of wavelength and frequency (k, w) detects electromagnetic waves of wavelength and frequency $(k', \frac{w'}{C}) = L(-u)(k, \frac{w}{C})$.*

Remark 4.7. The symmetry (4.3.29) corresponds to expressing the action in terms of the inferred charge and currents by an observer. As such, it corresponds to a subjective view of EM.

Remark 4.8. The Galilean variation that allowed us to obtain the Lorentz force from the action, eq. (4.3.31), indicates that the force experienced by the moving circuit takes the same form but the potentials to be used correspond to the perceived potentials of eq. (4.3.19).

4.3.4. Mathematical presentation of the Lorentz transformation as a symmetry. Since Lorentz transformations are well known in relation to electromagnetism, we consider their effect on the action and find their meaning in the present context.

Let $\mathcal{I}$ be the infinitesimal generator for the Lorentz transformation

$$\mathcal{I}_j = \left(Ct\frac{\partial}{\partial x_j} + \frac{x_j}{C}\frac{\partial}{\partial t} \right) \tag{4.3.29}$$

which together with the generators of the rotations

$$\mathcal{J}_i = \sum_{jk} \epsilon_{jki} \left(x_k \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_k} \right)$$

(with ϵ_{jki} Kronecker's antisymmetric tensor) complete the Lie algebra of the Poincaré-Lorentz group [Gilmore, 1974].

THEOREM 4.4. *The electromagnetic action* $\mathcal{A}$ (4.3.6) *transforms into an equivalent action* $\mathcal{A}'$ *when the infinitesimal transformations*

$$\hat{\delta} = \sum_i (\delta\theta_i \mathcal{J}_i + \delta v_i \mathcal{I}_i)$$

operate on $(j, C\rho)_{1,2}$ *simultaneously and* $\frac{d\delta v_i}{dt} = 0$.

PROOF. The result follows from the observation that the kernel K in (4.3.13) commutes with the six generators as a result of Theorem (4.2), and that, integrating by parts in space and time the action of $\hat{\delta}$ over $(j, C\rho)_2$ can be seen as an action over $(j, C\rho)_1$ preceded by a negative sign, and then, both actions compensate to first order. Thus, the infinitesimal action of any element of the Lie algebra acting on both subsystems (primary and secondary) corresponds to the identity. We have that

$$\mathcal{A} = \mathcal{A}' + \mathcal{F}(t)$$

with $\mathcal{F}(t)$ a functional of the potentials and currents evaluated at the time t. Since all variations are considered to be zero at the extremes of the time-interval, $\mathcal{F}(t)$ contributes to zero to the variational calculation. In terms of their variations, $\mathcal{A}$ and $\mathcal{A}'$ are equivalent. See the Appendix (4.A.5) for the algebraic details. □

The requirement for δv to be constant in time is familiar to any one acquainted with Lorentz transformations. It is interesting to mention that in the present context this requirement can be lifted by defining the variation as

$$\hat{\delta}_a = \sum_i \left(\delta\theta_i \mathcal{J}_i + \delta v_i \mathcal{I}_i + \frac{1}{2C}\frac{d\delta v_i}{dt} x_i \right) \tag{4.3.30}$$

4.3.5. The Lorentz force. The origins of the Lorentz force can be traced back to Maxwell and what he called the *Electromotive intensity* [[598], eq. B, Maxwell, 1873]. Similarly, Lorentz [1892] referred to Maxwell's electrokinetic energy and potential energy [[630,631] and [634,635], Maxwell, 1873], combining them in the action integral eq. (4.3.6). These presentations involved the ether some way or the other in the argumentation: Maxwell when considering the "total current" of eq. (4.3.8) and Lorentz in the variational principle.

Nevertheless, it is possible to perform an ether-free derivation of the Lorentz force [Natiello and Solari, 2021] from the Principle of Least Action for eq. (4.3.6). Let $\bar{x}(t)$ denote the distance between a reference point in the source and a reference point in the receiver. Recall that we label with 1 quantities belonging to the emitter (sender, source) and by 2 those of the target (probe, receiver). We consider, following Lorentz, a collection of virtual displacements $\delta\bar{x}(t)$ parameterised by time[53]. The variation of charge and current densities $\rho_2(y,t), j_2(y,t)$ on the receiver can be expressed in the coordinates of eq. (4.3.1) as:

$$\begin{aligned} \delta\rho_2(x,t) &= (-\delta\bar{x}(t)\cdot\nabla)\,\rho_2(x,t) \\ \delta j_2(x,t) &= (-\delta\bar{x}(t)\cdot\nabla)\,j_2(x,t) + \delta\dot{\bar{x}}\rho_2(x,t) \end{aligned} \tag{4.3.31}$$

The action integral in the present form corresponds to an S-by-S-field representation, namely that the fields of the source are evaluated at the position of the receiver in the coordinates x of the source and time t. We state the result as a theorem:

THEOREM 4.5. *[Natiello and Solari, 2021] Assuming that all of $|B|^2$,$|E|^2$, A, j, V, ρ decrease faster than $\frac{1}{r^2}$ at infinity, and that the action is given by eq. (4.3.6) and given the validity of the continuity equation, eq. (4.3.9), the electromagnetic force*

$$F_{em} = \int d^3x\,[j_2(x,t)\times B_1(x,t) + \rho_2(x,t)\,E_1(x,t)]$$

on the target (probe,receiver) can be deduced from Hamilton's principle of minimal action ($\delta_{\bar{x}(t)}\mathcal{A}=0$) using the virtual displacement $\delta_{\bar{x}}$ of the target with respect to the source producing the fields as in eq. (4.3.31).

The proof is in Appendix 4.A.4.

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[53] As in the Lagrangian formulation, the collection of virtual displacements is differentiable, i.e., $\dot{\bar{x}}$ exists, and the variation is zero in the borders of the time interval. Virtual displacements are not the same as time-dependent perturbations of the position, for the latter have other effects apart from the change of relative distances. Virtual displacements are closer to changes of initial conditions than to perturbations. In particular, during a virtual displacement, there is no wave progression.

In the case of Lorentz virtual displacements represented at the same time a displacement of the particle with respect to the ether and a displacement with respect to all other electromagnetic bodies present. In our case it represents only the latter which is labelled with the index 1.

In this presentation we consider the secondary circuit in motion while the primary circuit is the reference and can be thought of as the electromagnetic universe otherwise "at rest". According to NAP, there is an equivalent presentation attributing all the motion to the primary circuit, see below.

COROLLARY 4.3. *The force associated to the relative coordinate becomes:*

$$F = \frac{m_1 F^{1\to 2} - m_2 F^{2\mapsto 1}}{m_1 + m_2}.$$

For a proof, see the coming Subsection.

4.3.6. About inertial frames, centre of mass and action-reaction. When the source/receiver symmetry (see Subsection 4.3.2) is taken into account, the equivalent calculation corresponds to use as reference the system labelled 2 and attributing the motion to the system labelled 1. In such a case all the operations are performed on advanced potentials. The result is then a Lorentz force that acts upon system 1 with fields $\left(\tilde{E}, \tilde{B}\right)$ derived from the potentials $\left(\tilde{A}, \frac{\tilde{V}}{C}\right)$, eq. (4.3.3) . If the delay in the action can be neglected, Newton's third law (action equals reaction) is satisfied.

Indeed, the action integral for the interaction between systems 1 and 2 to be considered for the Lorentz force stems from the electromagnetic energy $\mathcal{E} = \frac{1}{2}\int d^3x \left[\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right]$ being the specific interaction part

$$\mathcal{A} = \int dt\, d^3x \left[\frac{1}{\mu_0} B_1 \cdot B_2 - \epsilon_0 E_1 \cdot E_2\right]. \tag{4.3.32}$$

Potentials and fields are related by Maxwell equations, eq. (4.3.4) and (4.3.5) while potential and current-charge are linked by the wave equation, Lemma 4.1. By Lemma 4.2 the action to be varied reads, in terms of potentials, charge and current:

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\int dr \int d^3x \left[A_1 \cdot j_2 - V_1 \rho_2 + A_2 \cdot j_1 - V_2 \rho_1\right] \\
\mathcal{A} &= \mathcal{A}^{1\mapsto 2} + \mathcal{A}^{2\mapsto 1} \\
\mathcal{A}^{1\mapsto 2} &= \frac{1}{2}\int dr \int d^3x \left[A_1 \cdot j_2 - V_1 \rho_2\right] \\
\mathcal{A}^{2\mapsto 1} &= \frac{1}{2}\int dr \int d^3x \left[A_2 \cdot j_1 - V_2 \rho_1\right]
\end{aligned}$$

while the variations, referred to varying charge and current are

$$\begin{aligned}
\Box \delta V_i &= -\frac{\delta \rho_i}{\epsilon_0} \\
\Box \delta A_i &= -\mu_0 \delta j_i \\
\delta B_i &= \nabla \times \delta A_i \\
\delta E_i &= -\nabla \delta V_i - \partial_t \delta A_i.
\end{aligned}$$

As customary, we are using retarded potentials throughout, but the symmetry with an advanced description becomes evident when expanding explicitly the contributions to the action integral. We write down one of them, the other is similar, recalling that $K(x-y,s-r) = \frac{1}{|y-x|}\delta(s-r-\frac{1}{C}|y-x|)$:

$$\begin{aligned}\mathcal{A}^{1\mapsto 2} &= \frac{\mu_0}{8\pi}\int d^3x\, d^3y\, drds\, \Big[j_2(x,r)\cdot j_1(y,s) - C^2\rho_2(x,r)\cdot\rho_1(y,s)\Big]\times \\ &\quad K(x-y,s-r) \\ &= \frac{1}{2}\int d^3x\, dr\, [j_2(x,r)\cdot A_1(x,r) - \rho_2(x,r)\cdot V_1(x,r)] \\ &= \frac{1}{2}\int d^3y\, ds\, \Big[\tilde{A}_2(y,s)\cdot j_1(y,s) - \tilde{V}_2(y,s)\cdot\rho_1(y,s)\Big] = \tilde{\mathcal{A}}^{2\mapsto 1}. \quad (4.3.33)\end{aligned}$$

Applying Lemma 4.2 again, we recover the corresponding expression of eq. (4.3.32) in terms of advanced potentials.

Concerning the use of the least action principle, in Newtonian mechanics the action and reaction principle assured that the total force on the centre of mass was zero, and the force acting on the relative coordinate was $\frac{m_1F^{1\to 2}-m_2F^{2\mapsto 1}}{m_1+m_2} = F^{1\mapsto 2}$ (up to a sign). In the present situation there is no action and reaction principle since the forces are not of instantaneous action. However, inspired in the limit $m_1 \gg m_2$ we regard the expressions of charge and current as given in the centre of mass (CM) system. Hence, the only meaningful variation is to consider the relative displacement $\delta\bar{x}$ of the electromagnetic entities, distributed as $\delta\bar{x}_1 = -\frac{m_2}{m_1+m_2}\delta\bar{x}$ and $\delta\bar{x}_2 = \frac{m_1}{m_1+m_2}\delta\bar{x}$, yielding $\delta\bar{x}_2 - \delta\bar{x}_1 = \delta\bar{x}$ and $\delta\bar{x}_{CM} = \frac{m_1\delta\bar{x}_1+m_2\delta\bar{x}_2}{m_1+m_2} = 0$. We regard the action as written in terms of the relative coordinate. In this way, the action integral does not depend on $\bar{x}_{CM}$(at any time) and the only valid variation corresponds to the relative coordinate.

The variation of the action with respect to the relative coordinate has a term associated to each system (which in this case cannot be thought of as primary and secondary since both play both roles). The resulting force is:

$$F = \frac{m_1F^{1\to 2} - m_2F^{2\mapsto 1}}{m_1+m_2},$$

thus proving corollary 4.3.

Remark 4.9. Action and reaction. With respect to action and reaction it is interesting to observe that once we have chosen that the primary system, 1, is acting on the secondary system, 2, the form of the action is defined as in eq. (4.3.33). If we consider for example that the primary system is fixed to the laboratory it is possible to obtain the mechanical force associated to the constraint as a variation of the constraint equation, in this case it means to change rigidly the position of the primary by an amount $-\delta x$ at a given time t. The calculation requires to use the form of the action in terms of advanced potentials thus obtaining

$$\tilde{F}^{1\mapsto\,2} = -\int d^3x\, \left\{\rho_1\tilde{E}_2 + j_1\times\tilde{B}_2\right\}$$

thus $\tilde{F}^{1\mapsto\,2}$ is the contribution to the reaction due to the action of 1 on 2. Since in the limit where we can consider $C\to\infty$ $\left(\tilde{E}_2,\tilde{B}_2\right)\to(E_2,B_2)$ an action-reaction principle is recovered. $\tilde{F}^{1\mapsto\,2}$ represents the contribution to the effort put on the fixation that corresponds to the action on system 2. The total reaction would require to add up all the intervening circuits (say the universe) and there is yet a question on what would happen if some waves never interact with other matter.

Appendix

4.A. Some Proofs

4.A.1. Proof of Lemma 4.1.

Proof. We perform the calculation in detail only for A, since the other one is similar. We use the shorthand $r = |x - y|$.

$$\begin{aligned}
\nabla_x A_i &= \frac{\mu_0}{4\pi}\int d^3y\left(j_i \nabla_x \frac{1}{r} - \frac{\frac{\partial}{\partial t} j_i \nabla_x \frac{r}{C}}{r}\right) \\
\Delta A_i &= \nabla_x \cdot \nabla_x A_i \\
&= \frac{\mu_0}{4\pi}\int d^3y\left(j_i \Delta \frac{1}{r} - 2\left(\nabla_x \frac{1}{r}\right)\cdot\left(\frac{\partial}{\partial t} j_i \nabla_x \frac{r}{C}\right) - \frac{\frac{\partial}{\partial t} j_i \Delta \frac{r}{C}}{r} + \frac{\frac{\partial^2}{\partial t^2} j_i}{r}|\nabla_x \frac{r}{C}|^2\right)
\end{aligned}$$

Moreover, standard vector calculus identities give

$$\begin{aligned}
\frac{\partial}{\partial t} j_i\left(2\nabla\frac{1}{r}\cdot\nabla\frac{r}{C} + \frac{\Delta\frac{r}{C}}{r}\right) &= 0 \\
|\nabla \frac{r}{C}|^2 &= \frac{1}{C^2}
\end{aligned}$$

and therefore

$$\Delta A_i(x,t) = \frac{\mu_0}{4\pi}\int d^3y\, j_i(y, t - \frac{r}{C})\Delta\left(\frac{1}{r}\right) + \left(\frac{1}{C^2}\right)\frac{\mu_0}{4\pi}\int d^3y\,\frac{\partial^2}{\partial t^2}\frac{j_i(y, t-\frac{r}{C})}{r}$$

The time derivative in the last term can be extracted outside the integral, thus yielding,

$$\begin{aligned}
\Box A_i(x,t) &= \Delta A_i(x,t) - \left(\frac{1}{C^2}\right)\frac{\mu_0}{4\pi}\int d^3y\,\frac{\partial^2}{\partial t^2}\frac{j_i(y, t-\frac{r}{C})}{r} \\
&= \Delta A_i(x,t) - \left(\frac{1}{C^2}\right)\frac{\partial^2}{\partial t^2}A_i(x,t) \\
&= \frac{\mu_0}{4\pi}\int d^3y\, j_i(y, t - \frac{|x-y|}{C})\Delta\left(\frac{1}{r}\right) \\
&= -\mu_0 j_i(x,t)
\end{aligned}$$

□

4.A.2. Proof of Theorem 4.1.

PROOF. The result follows from the computation of the extremal action under the constraints

$$\begin{aligned}(V-\mathsf{V})\chi &= 0\\ (A-\mathsf{A})\chi &= 0\end{aligned}$$

Multiplying the constraints by the Lagrange multipliers λ and κ (the latter a vector), while we use the shorthand notations $B=\nabla\times A$ and $E=\left(-\frac{\partial A}{\partial t}-\nabla V\right)$, we need to variate the constrained electromagnetic action

$$\mathcal{A}=\frac{1}{2}\int dt\left(\int\left(\frac{1}{\mu_0}|B|^2-\epsilon_0|E|^2-\kappa\cdot(A-\mathsf{A})\chi+\lambda(V-\mathsf{V})\chi\right)d^3x\right).$$

Varying the integrand we obtain

$$\begin{aligned}\delta\mathcal{A} &= \int dt\left(\int\left(\frac{1}{\mu_0}(\nabla\times A)\cdot(\nabla\times\delta A)-\epsilon_0\left(\nabla V\cdot\nabla\delta V+\frac{\partial A}{\partial t}\frac{\partial\delta A}{\partial t}\right)\right.\right.\\ &\quad\left.\left.-\epsilon_0\left(\nabla V\cdot\frac{\partial\delta A}{\partial t}+\nabla\delta V\cdot\frac{\partial A}{\partial t}\right)-\chi\kappa\,\delta A+\chi\lambda\,\delta V\right)d^3x\right)\end{aligned}$$

Partial integrations in time and standard vector calculus give the following identities:

$$\begin{aligned}\int dt\,\frac{\partial A}{\partial t}\frac{\partial\delta A}{\partial t} &= \left[\delta A\cdot\frac{\partial A}{\partial t}\right]-\int dt\,\delta A\cdot\frac{\partial^2 A}{\partial t^2}\\ \int dt\,\nabla V\cdot\frac{\partial\delta A}{\partial t} &= [\delta A\cdot\nabla V]-\int dt\,\delta A\cdot\nabla\frac{\partial V}{\partial t}\\ (\nabla\times A)\cdot(\nabla\times\delta A) &= \nabla\times(\nabla\times A)\cdot\delta A-[\nabla\cdot((\nabla\times A)\times\delta A)]\\ \left(\nabla V+\frac{\partial A}{\partial t}\right)\cdot\nabla\delta V &= \left[\nabla\cdot\left(\left(\nabla V+\frac{\partial A}{\partial t}\right)\delta V\right)\right]-\delta V\nabla\cdot\left(\nabla V+\frac{\partial A}{\partial t}\right)\end{aligned}$$

The terms in square brackets vanish in the variation either for the vanishing variation at endpoints or because of Gauss theorem applied to functions decaying fast enough at infinity. Hence,

$$\begin{aligned}\delta\mathcal{A} &= \int dt\int d^3x\left(\frac{1}{\mu_0}(\nabla\times A)\cdot(\nabla\times\delta A)-\epsilon_0\left(-\delta V\nabla\cdot\left(\nabla V+\frac{\partial A}{\partial t}\right)\right.\right.\\ &\quad\left.\left.-\delta A\cdot\frac{\partial^2 A}{\partial t^2}-\delta A\cdot\nabla\frac{\partial V}{\partial t}\right)-\chi\kappa\cdot\delta A+\chi\lambda\delta V\right).\end{aligned}$$

Being δA and δV independent, we obtain

$$\begin{aligned}\frac{1}{\mu_0}\nabla\times(\nabla\times A)+\epsilon_0\left(\frac{\partial^2 A}{\partial t^2}+\nabla\frac{\partial V}{\partial t}\right) &= \chi\kappa\\ -\epsilon_0\nabla\cdot\left(\nabla V+\frac{\partial A}{\partial t}\right) &= \chi\lambda\end{aligned}$$

or equivalently

$$\begin{aligned} \frac{1}{\mu_0}\nabla\times B-\epsilon_0\frac{\partial E}{\partial t} &= \chi\kappa \\ \epsilon_0\nabla\cdot E &= \chi\lambda \end{aligned}$$

which allows us to identify $j=\chi\kappa$ (the density of current inside the material responsible for A) and $\rho=\chi\lambda$ (the density of charge responsible for V), thus proving the first result. Finally, the continuity equation follows from

$$0=\nabla\cdot\left(\frac{1}{\mu_0}\nabla\times B-\epsilon_0\frac{\partial E}{\partial t}\right)+\frac{\partial}{\partial t}\left(\epsilon_0\nabla\cdot E\right)=\nabla\cdot j+\frac{\partial\rho}{\partial t}.$$

Note also that taking curl on the first equation we verify that B satisfies a wave equation. Inserting $\nabla\times E$ in the time-derivative of the first equation and adding the gradient of the second equation, we obtain a wave equation for E.

Further, $\frac{1}{\mu_0}\nabla\cdot A+\epsilon_0\frac{\partial V}{\partial t}=0$ implies both that $\nabla\frac{\partial V}{\partial t}=-C^2\nabla\left(\nabla\cdot A\right)$ and $\nabla\cdot\frac{\partial A}{\partial t}=-\frac{1}{C^2}\frac{\partial^2 V}{\partial t^2}$. Substituting each relation in the corresponding equation, we obtain eq((4.3.2)), thus proving the Corollary. □

4.A.3. Proof of Lemma 4.4.

Proof. Under the general assumption that $\int\nabla\cdot F(x,t)\,d^3x$ vanishes at infinity, being F a vector function that decays sufficiently fast (i.e., faster than r^{-2}), we obtain

$$\begin{aligned} &\int_0^t dt\int d^3x\left(\left(\nabla V+\frac{\partial A}{\partial t}\right)\cdot j\right)= \\ =&\int_0^t dt\int d^3x\left(\nabla\cdot(Vj)-V\nabla\cdot j+\frac{\partial A}{\partial t}\cdot\left(-\epsilon_0\frac{\partial E}{\partial t}+\frac{1}{\mu_0}\nabla\times B\right)\right) \\ =&\int_0^t dt\int d^3x\left(V\frac{\partial\rho}{\partial t}+\frac{\partial A}{\partial t}\cdot\left(-\epsilon_0\frac{\partial E}{\partial t}+\frac{1}{\mu_0}\nabla\times B\right)\right) \\ =&\int_0^t dt\int d^3x\left(\epsilon_0 V\nabla\cdot\frac{\partial E}{\partial t}-\epsilon_0\frac{\partial A}{\partial t}\cdot\frac{\partial E}{\partial t}+\frac{1}{\mu_0}\frac{\partial A}{\partial t}\cdot\nabla\times B\right) \\ =&\int_0^t dt\int d^3x\left(\epsilon_0\nabla\cdot(V\frac{\partial E}{\partial t})-\epsilon_0\nabla V\cdot\frac{\partial E}{\partial t}-\epsilon_0\frac{\partial A}{\partial t}\cdot\frac{\partial E}{\partial t}+\frac{1}{\mu_0}\left(\frac{\partial B}{\partial t}\cdot B\right)\right. \\ &\left.-\frac{1}{\mu_0}\nabla\cdot\left(\frac{\partial A}{\partial t}\times B\right)\right) \\ =&\int_0^t dt\int d^3x\left(\epsilon_0 E\cdot\frac{\partial E}{\partial t}+\frac{1}{\mu_0}\left(\frac{\partial B}{\partial t}\cdot B\right)\right) \\ =&\frac{1}{2}\int_0^t dt\frac{\partial}{\partial t}\int d^3x\left(\epsilon_0|E|^2+\frac{1}{\mu_0}|B|^2\right) \\ =&\frac{1}{2}\int d^3x\left(\epsilon_0|E|^2+\frac{1}{\mu_0}|B|^2\right) \end{aligned}$$

where all integrals involving total divergences have been set to zero by Gauss' theorem. We have also used the continuity equation. □

4.A.4. Deduction of Lorentz force revisited. When we consider two pieces of electrified matter in interaction we can envisage a different form of constructing the system. In the first step, the bodies are far apart, so that we can assume that they do not interact, and are electrified to reach their actual state. Next, they will be brought together to their corresponding mechanical positions in terms of a thought process called a *virtual displacement*. The formalisation of this idea already present in Maxwell is called a *virtual variation* [Ch 4, 21 B p. 92, Arnold, 1989]. The force associated to this virtual displacement will then result from the variation of the interaction terms in the Lagrangian. Given the electromagnetic contribution to the action, determining the contribution to the force amounts to applying Hamilton's principle using a virtual displacement of the probe with respect to the source producing the fields. In formulae, we must request

$$\delta_{\bar{x}(t)}\mathcal{A} = 0$$

with $\delta\bar{x}(t_0) = 0$ and $\delta\bar{x}(t) = 0$, where $\bar{x}(t)$ denotes the relative distance between probe and source. In so doing, we must take into account that there is a corresponding variation of the velocity $\delta\dot{\bar{x}}(t)$. Recall that according to (4.3.31) $\bar{x}$ occurs in (ρ_2, j_2) and $\dot{\bar{x}}$ occurs only in j_2. We want to prove Theorem 4.5.

Proof. We display in what follows only the electromagnetic part of the action. To derive the dynamical equations, the kinetic energy associated to the movement has to be taken into account in the action integral. The present proof focus on the "force" side. In terms of charged point particles, the variation of the kinetic energy gives $\frac{m_1 m_2}{m_1 + m_2}\frac{d^2}{dt^2}(x_2 - x_1)$.

For the sake of convenience, we may express the action using Lemma 4.2 before proceeding:

$$\delta_{\bar{x}(t)}\mathcal{A} = \delta_{\bar{x}(t)}\frac{1}{2}\int dt \int d^3x\ [A\cdot j - V\rho] = \frac{1}{2}\int dt \int d^3x\,[\delta A\cdot j + A\cdot\delta j - \delta V\rho - V\delta\rho] \tag{4.A.1}$$

Potentials, charge and current can be split into primary and secondary circuits, namely $A = A_1 + A_2$ (and similarly for V, j, ρ) and the variation can be formulated as moving the secondary circuit with respect to a fixed primary circuit. Only the interacting terms actually vary, so the integrand reads $\frac{1}{2}(\delta A_2 \cdot j_1 - \delta V_2\rho_1) + \frac{1}{2}(A_1\cdot\delta j_2 - V_1\delta\rho_2)$. We transform the first bracket using Maxwell equations, Lemma 4.1:

$$\frac{1}{2}\int dt\int d^3x\ [\delta A_2\cdot j_1 - \delta V_2\rho_1] \quad = \quad \frac{1}{2}\int dt\int d^3x\,\left[-\frac{1}{\mu_0}\delta A_2\cdot\Box A_1 + \epsilon_0\delta V_2\Box V_1\right]$$

The particular variation reflecting the problem (moving subsystem 2 with respect to a fixed subsystem 1) and partial integrations in space and time allow to transform

$$\frac{1}{2}\int dt \int d^3x \, [\delta A_2 \cdot j_1 - \delta V_2 \rho_1] = \frac{1}{2}\int dt \int d^3x \left[-\frac{1}{\mu_0}\Box\delta A_2 \cdot A_1 + \epsilon_0 \Box \delta V_2 V_1\right]$$

up to an overall divergence whose contribution vanishes by Gauss theorem and an overall function of time that does not contribute to the variation. Hence, the required variation reads (dropping the index $\bar{x}(t)$),

$$\delta\mathcal{A} = \int dt \int d^3x \, [A_1 \cdot \delta j_2 - V_1 \delta\rho_2]$$

The variation of the current and of the charge distribution due to the motion of the probe relative to the primary circuit are, after eq. (4.3.31),

$$\begin{aligned} \delta j_2 &= -(\delta\bar{x}\cdot\nabla) j_2 + \rho_2 \delta\dot{\bar{x}} \\ \delta\rho_2 &= -(\delta\bar{x}\cdot\nabla)\rho_2 \end{aligned} \tag{4.A.2}$$

We have then

$$\begin{aligned} \delta\mathcal{A} &= \delta \int dt \, \mathcal{L} = \int dt \int d^3x \, [A_1 \cdot \delta j_2 - V_1 \delta\rho_2] \\ &= \int dt \int d^3x \, \left[A_1 \cdot \left(-(\delta\bar{x}\cdot\nabla) j_2 + \rho_2\delta\dot{\bar{x}}\right) - V_1 \left(-(\delta\bar{x}\cdot\nabla)\rho_2\right)\right] \\ &= \int dt \int d^3x \left[j_2 \cdot (\delta\bar{x}\cdot\nabla) A_1 - \rho_2 (\delta\bar{x}\cdot\nabla) V_1 - \delta\bar{x}\cdot\frac{\partial}{\partial t}(A_1\rho_2)\right] \end{aligned} \tag{4.A.3}$$

The last line is obtained after some integrations by parts and following the cancellation of a whole divergence via Gauss theorem and of an overall function of time via the variational constraints. In particular:

$$\begin{aligned} ((\delta\bar{x}\cdot\nabla) j_2)\cdot A_1 &= (\delta\bar{x}\cdot\nabla)(j_2\cdot A_1) - j_2\cdot(\delta\bar{x}\cdot\nabla) A_1 \\ &= \nabla\cdot(\delta\bar{x}\,(j_2\cdot A_1)) - j_2\cdot(\delta\bar{x}\cdot\nabla) A_1 \\ V_1(\delta\bar{x}\cdot\nabla)\rho_2 &= \delta\bar{x}\cdot\nabla(V_1\rho_2) - \rho_2(\delta\bar{x}\cdot\nabla) V_1 \\ &= \nabla\cdot(\delta\bar{x} V_1\rho_2) - \rho_2(\delta\bar{x}\cdot\nabla) V_1 \\ A_1\cdot\left(\rho_2\delta\dot{\bar{x}}\right) &= \frac{\partial}{\partial t}(\rho_2 A_1\cdot\delta\bar{x}) - \delta\bar{x}\cdot\frac{\partial}{\partial t}(A_1\rho_2) \end{aligned}$$

Further transformations with mathematical identities allow us to write

$$\begin{aligned} -\rho_2(\delta\bar{x}\cdot\nabla) V_1 - \rho_2\delta\bar{x}\cdot\frac{\partial}{\partial t}A_1 &= \delta\bar{x}\cdot\rho_2 E_1 \\ j_2\cdot(\delta\bar{x}\cdot\nabla) A_1 - \delta\bar{x}\cdot A_1\frac{\partial}{\partial t}\rho_2 &= j_2\cdot(\delta\bar{x}\cdot\nabla) A_1 + (\delta\bar{x}\cdot A_1)(\nabla\cdot j_2) \\ &= j_2\cdot(\delta\bar{x}\cdot\nabla) A_1 + \nabla\cdot(j_2(\delta\bar{x}\cdot A_1)) \\ &\quad - j_2\cdot\nabla(\delta\bar{x}\cdot A_1) \\ &= \nabla\cdot(j_2(\delta\bar{x}\cdot A_1)) - j_2\cdot\delta\bar{x}\times(\nabla\times A_1) \\ &= \nabla\cdot(j_2(\delta\bar{x}\cdot A_1)) + \delta\bar{x}\cdot j_2\times(\nabla\times A_1) \end{aligned}$$

and finally, after applying Gauss theorem again,

$$\int dt \int d^3x\, \delta\bar{x} \cdot [j_2 \times B_1 + \rho_2 E_1]\,.$$

This is, following the standard use of Hamilton's principle in mechanics we arrive to an electromagnetic contribution to the force

$$F_{em} = \int d^3x\ (j_2 \times B_1 + \rho_2 E_1)$$

□

4.A.5. Proof of Theorem 4.4.

PROOF. The rotational invariance is immediate, since for any rotation matrix R,the change of coordinates $x' = Rx$ (along with the corresponding change for y), keeps the distance $|R(x-y)| = |x-y|$ invariant. Hence, for the kernel in eq. (4.3.13) and any electromagnetic kernel depending on $|x-y|$ the action integral is invariant under rotations. Let u be the velocity associated to a Lorentz transformation, which is constant by hypothesis. The proposed variation reads

$$\begin{aligned}\hat{\delta}\mathcal{A} &= \int_{-\infty}^{t} ds \int d^3x\, \hat{\delta}\left[A^1(x,s) j_2(x,s) - V^1(x,s)\rho_2(x,s)\right] \\ &= \int_{-\infty}^{t} ds \int d^3x \left(Cs\,\delta u \cdot \nabla_x + (\frac{x}{C}\cdot \delta u)\partial_s\right)\left[A^1 j_2 - V^1 \rho_2\right](x,s)\end{aligned}$$

where $\mathcal{I}(x,s) = Cs\,\delta u \cdot \nabla_x + (x \cdot \delta u)\partial_{Cs}$ is the Lorentz generator. By Gauss Theorem the following integral vanishes for any function F inheriting the behaviour of A, j at infinity:

$$\int_K d^3x\, \delta u \cdot \nabla F(x) \quad = \quad \int_{\partial K} F(x)\delta u \cdot dS = 0$$

Finally,

$$\int_{-\infty}^{t} ds\, \frac{\partial}{\partial s} \int d^3x\, (x\cdot \delta u) G(x,s) \quad = \quad \mathcal{F}(t) - \mathcal{F}(t_0)$$

for some function $\mathcal{F}$ depending only of t. However by the nature of the variational process, $\mathcal{F}$ does not contribute to the variation. □

4.B. The Lorentz transformation

The infinitesimal generator of the Lorentz transformation in eq. (4.3.20) reads

$$\mathcal{I}_j = \begin{pmatrix} \mathbf{0} & \frac{v}{C} \\ (\frac{v}{C})^T & 0 \end{pmatrix}.$$

The Lorentz transformation for finite v is obtained by exponentiation [Gilmore, 1974], yielding the 4×4 matrix expression $TL(v)$ for the transformation elements, where

$$TL(v) = \begin{pmatrix} W & X \\ X^{\dagger} & Y \end{pmatrix}$$

is formed by the 3-vector $X = \sinh\left(\left|\frac{v}{C}\right|\right)\frac{v}{|v|}$, the scalar $Y = \sqrt{1+|X|^2} = \cosh\left(\left|\frac{v}{C}\right|\right)$ and the 3×3 matrix $W = Id + \left(\cosh\left(\left|\frac{v}{C}\right|\right) - 1\right)\frac{vv^\dagger}{|v|^2}$, where $v \in \mathbb{R}^3$ is a parameter classifying the different transformations. A better known expression for the Lorentz transformation arises from the change of variables $u = C\hat{v}\tanh\left|\frac{v}{C}\right|$ [Gilmore, 1974]. In such terms,

$$L(u) = \begin{pmatrix} Id + (\gamma-1)\hat{u}\hat{u}^\dagger & \gamma\frac{u}{C} \\ \gamma\frac{u^\dagger}{C} & \gamma \end{pmatrix}; \quad L\left(u(v)\right) \equiv TL(v),$$

where $\gamma\left(u(v)\right) = \dfrac{1}{\sqrt{1-\left|\frac{u(v)}{C}\right|^2}} = \cosh\left(\left|\frac{v}{C}\right|\right)$. Note that for $\left|\frac{v}{C}\right| \ll 1$, we have that $\left|\frac{u}{C}\right| = \left|\frac{v}{C}\right| + O\left(\left|\frac{v}{C}\right|^3\right)$.

4.C. Scholium. Kaufmann's experiment

4.C.1. The experiment. In Kaufmann's experiment [Kaufmann, 1906] a beam of high-speed electrons is deflected under the influence of electrostatic and magnetostatic fields (see Figure 4.C.1).The velocity u is measured as in Thomson's experiment balancing the electric and magnetic force. According to Lorentz' law of force,

$$u = \frac{E}{B}$$

where the electric field E and the magnetic field B are perpendicular and perpendicular to the velocity as well. To fix ideas we write

$$\begin{aligned} \mathbf{u} &= (0,0,u) \\ \mathbf{E} &= (E,0,0) \\ \mathbf{B} &= (0,B,0) \\ \mathbf{F} &= q(\mathbf{E} + \mathbf{u}\times\mathbf{B}) = q(E-uB,0,0) \end{aligned}$$

When only one of the fields is applied, the charge experiences a force and then a deviation, being the measured quantity the angle of deviation ϕ, using Newton's mechanics. If the transit time through the apparatus is $\Delta t = \frac{L}{|u|}$, with L the longitudinal effective dimension and only the magnetic field is acting, the particle will acquire a momentum in direction $\hat{\mathbf{1}}$,

$$-quB\Delta t = F\Delta t$$

Therefore, the tangent of the deviation angle ϕ relative to the velocity u will be

$$|\tan(\phi)| = \frac{|F\Delta t|}{m}\frac{1}{|u|} = \frac{|q|}{m}|B|\frac{L}{|u|} = |q||B|\frac{L|B|}{m|E|} \tag{4.C.1}$$

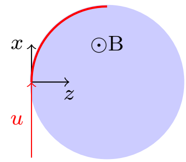

FIGURE 4.C.1. Sketch of Kaufmann experiment

Hence, $m = \dfrac{|q||B|^2 L}{|E||\tan(\phi)|)}$. Plotting m as a function of $|u| = \dfrac{|E|}{|B|}$, specifying the values of the fields as those measured in the lab, it is found that the mass is compatible with $m = \dfrac{m_0}{\sqrt{1-(u/C)^2}} + o\left(\dfrac{|u|^2}{C^2}\right)$.

A sort of myth emerges at this point. Lorentz [Lorentz, 1904] tried to fix the electromagnetism conceived by consideration of the ether. After lengthy mathematical manipulations of Maxwell's equations he arrived to the point in which the force exerted in the direction of motion held a different relation with the change of mechanical momentum than the force exerted in the perpendicular direction. He then substituted a constant velocity introduced as a change of reference frame with the time-dependent velocity of the electron with respect to the ether. This is a typical utilitarian approach, the formula is applied outside its scope although the mathematics that led to it does not hold true if the velocity is not constant. This is the origin of Lorentz' mass formula as applied for example to Kaufmann's experiment. The same utilitarian operation is used to produce the relativistic mass formulae. However, if this new mass were to be used in the derivation of Lorentz' force [Lorentz, 1892], the usual expression of the force would not hold. The story illustrates well the "advantages" of the utilitarian approach: it frees the scientist of requirements of consistency. In contrast, relational electromagnetism acknowledges that when in motion relative to the source, we perceive EM fields still as EM fields but with characteristic changes as it is made evident by the Doppler effect earlier in this Chapter.

A number of experiments of varying accuracy have been performed along the years and reviewed in [Faragó and Jánossy, 1957]. Their goal was to decide between two functional dependencies for the variation of mass with velocity. According to Faragó & Jánossy, only one experiment [Glitscher, 1917] could offer results in favour of $m = \dfrac{m_0}{\sqrt{1-(u/C)^2}} + o\left(\dfrac{|u|^2}{C^2}\right)$, being the others undecidable. The experiment is based on the fine structure of Hydrogen-like atoms. Interpreting the experimental results requires an atomic model, in Glitscher's 1917 experiment it was Bohr's model. No experiment can determine a specific functional dependency among the myriad of functions that are compatible with the measurements.

4.C.2. Criticism of the velocity dependent mass interpretation. The experimental results relate $\tan(\phi)$ with $\dfrac{u}{C}$ while the interpretation of the results is based upon two bold assumptions. First, that the motion is governed by Newton's law (h1) and second that the Lorentz' force involved is derived from the EM fields produced and perceived in the laboratory (h2). Then, the experiment refutes the statement "h1 & h2 is true". Despite the experimental evidence that h2 is false given by the Doppler effect, it has been conventionally held that h1 must be declared refuted by the experiment. It turns to be that, when the Lorentz force is derived in accordance with the fields perceived by the moving charge that accounts for the Doppler effect (see Solari and Natiello [2022b]), the resulting transversal force must be scaled by a factor $\dfrac{1}{\gamma}$.

The dependence of the mass with velocity is only a patch for the wrong hypothesis h2. We address this matter in the next subsection.

4.C.3. Calculation of the force. When Hamilton's principle is used to deduce the force on a pointlike particle, a phenomenological step must be taken. The step consists in identifying on one side the current-charge associated to the particle, and on the other side the EM fields. There are not too many intuitive current-charges that could be associated to arbitrary trajectories of a point charge. This "technical" limitation makes necessary the use of

$$\begin{pmatrix} qv \\ qC \end{pmatrix} \delta(x - \bar{x})$$

which is an intuited form based upon experiences with small values of $\frac{|v|}{C}$. Actually, the charge of the electron was first determined by Millikan using charges slowly moving with respect to the laboratory and balancing gravitational forces. Thus, the current-charge above has experimental support under the condition that $|v| \ll C$, it would make sense in a frame where the charge is approximately at rest but says little about the case used for the analysis of the experiment.

For the EM field we know that the characteristics of the perceived EM action differ depending on the relative velocity of the detector with respect to the source, as it has been precisely shown in measurements of the Doppler effect at high speeds [Mandelberg and Witten, 1962, Kaivola et al., 1985]. Hence, the safe form of determining the Lorentz' force, without engaging in wild conjectures, is to use a frame which moves at a given instant with the same velocity than the particle with respect to the laboratory, call it u, so that the particle moves slowly (for a while) in this frame and we can more safely use our intuited charge and current[54]. Since, charge and fields are not in correspondence, to solve the issue we must find the appropriated field and move where the assumed current-charge makes sense. Thus, $v = \dfrac{d\bar{x}'}{dt} = \dfrac{d\bar{x}}{dt} - u$, where $\bar{x}$ is the trajectory described in the laboratory frame, $\bar{x}'$ is the trajectory described in the moving frame (where the charge is at rest at the mid point of the electron travelling between the laboratory fields) and $|v| \ll C$. In what follows we use the coordinates (x, y, z) with respect to the laboratory and

[54]This idea can be found as well in [§10 Dynamics of the Slowly Accelerated Electron, Einstein, 1905a].

(x', y', z') with respect to the moving frame, the transformation being $(x, y, z) = (x', y', z' + ut + z_0)$ (in what follows $z_0 = 0$ without loss of generality).

Before we proceed with the calculation we need to rewrite the action

$$\begin{aligned} \mathcal{A} &= \frac{1}{2}\int dt \int \left(A' \cdot j' - \rho' V'\right) d^3x' \\ &= \frac{1}{2}\int dt \int d^3x' \left((A', V'/C) \odot (j', C\rho')\right) \\ &= \frac{1}{2}\int dt \int d^3x \left((A, V/C) \odot (j, C\rho)\right) \end{aligned}$$

by Corollary 4.2 (the symbol $\odot$ stands for the product using Minkowski's metric).

We write $A = \frac{B}{2}(z, 0, -x)\,\Upsilon$, $V = -Ex\Upsilon$ for the potentials in the laboratory and $\mathbf{u} = (0, 0, u)$ for the velocity. The function Υ represents the restriction to a region of action which can be regarded as a cube of side L. We will disregard border effects (i.e., contributions of the derivatives of Υ to the fields). Next, the laboratory is put in relative motion transforming the potentials in terms of the perception in a situation of relative motion (see Subsubsection 4.3.3.1), thus

$$\begin{aligned} \left(A', \frac{V'}{C}\right) &= L(-u)(A, \frac{V}{C})(L(u)(x', t)) \\ &= \left(A_1, 0, \gamma\left(A_3 - \frac{u}{C^2}V\right), \frac{\gamma}{C}\left(V - uA_3\right)\right)\left(x', y', \gamma\left(z' + ut\right)\right) \end{aligned}$$

where the third and fourth components of the potentials arise through a Lorentz boost along the direction of $\mathbf{u}$, namely

$$\gamma \begin{pmatrix} 1 & -u/C \\ -u/C & 1 \end{pmatrix} \begin{pmatrix} A_3 \\ V/C \end{pmatrix}.$$

The charge-current is

$$\begin{pmatrix} j \\ \rho C \end{pmatrix}' (\mathbf{x}', t) = \begin{pmatrix} qv \\ qC \end{pmatrix} \delta(\mathbf{x}' - \bar{\mathbf{x}}')$$

The origin of time and the origin of coordinates has been chosen so that $\bar{\mathbf{x}}' = 0$ at $t = 0$. In our particular case the potentials read

$$\left(A', \frac{V'}{C}\right) = \left(\frac{B}{2}\gamma(z' + ut), 0, -\gamma x'\left(\frac{B}{2} - \frac{Eu}{C^2}\right), -\frac{\gamma}{C}x'\left(E - u\frac{B}{2}\right)\right)\Upsilon'$$

and $\Upsilon' = \Upsilon(x', y', \gamma(z' + ut))$. The Lorentz' variation takes the usual form

$$\begin{pmatrix} \delta j \\ \delta\rho C \end{pmatrix}' (\mathbf{x}', t) = \begin{pmatrix} v \\ C \end{pmatrix} q\left(\delta\bar{\mathbf{x}}' \cdot \nabla\right)\delta(\mathbf{x}' - \bar{\mathbf{x}}') + \begin{pmatrix} \delta v \\ 0 \end{pmatrix} q\delta(\mathbf{x}' - \bar{\mathbf{x}}')$$

while the force becomes

$$\begin{aligned} F &\sim \int d^3x \left[\rho\Upsilon'\left(-\nabla V' - A'_{,t} + v \times \left(\nabla \times A'\right)\right)\right] \\ &= q\left(\gamma\left(E - uB\right)\hat{x} + v \times B'\right)\Upsilon(\bar{x}', \bar{y}', \gamma(\bar{z}' + ut)) \end{aligned}$$

where $B' = \hat{y}\gamma(B - uE/C^2)\Upsilon(\bar{x}', \bar{y}', \gamma(\bar{z}' + ut))$ and the time Δt during which the force F is operative is

$$\Delta t \sim \left| \frac{L}{\gamma u} \right| \tag{4.C.2}$$

The exit velocity reads then $(\gamma u, F\Delta t)$, the equilibrium velocity is

$$u = \frac{E}{B}$$

and the tangent of the angle is

$$|\tan(\phi)| = \frac{|F\Delta t|}{m}\frac{1}{\gamma|u|} = \frac{|q|}{m}|B|\frac{L}{\gamma|u|} = \frac{1}{\gamma m}|q||B|\frac{L|B|}{|E|}$$

The calculation carried out shows that the component of the force transversal to the motion is weakened by a factor $1/\gamma$, implying that Kaufmann's experimental result is aligned with EM while the electromagnetically measured velocity remains $u = \dot{\bar{x}} = \dfrac{E}{B}$. The missing element in the original calculation used for the experiment is the change in the perception of the electromagnetic fields due to the charge being in motion with respect to the laboratory: the Doppler effect.

As a technical point, it is important to understand that the variational principle does not know whether a charge-current or a potential is being described in relation to the observer standing in the laboratory, or to an observer passing by or running along with the electron. Because of the perception symmetry of the action (Corollary 4.2) there is no preference in writing it in one or another frame, however the symmetry indicates that there is no electromagnetic knowledge of the velocity. If we can measure the velocity of a star by the Doppler shift of a recognisable atomic spectrum is precisely because we can compare the spectrum with the results of experiments made in the laboratory. The velocity results from the comparison of the EM signal with one produced at rest by the same (conjectured) system. In the same form, the Lorentz variation is a variation of the conditions at a given time of the location of the mass. In the case of a pointlike particle what is varied is $\bar{\mathbf{x}}(t)$. If we are dealing with the perceived charge-current the variation will evaluate in $\gamma\left(t - \frac{\mathbf{u}\cdot\mathbf{x}}{C^2}\right)$ and we will have the mathematical task of finding an expression in terms of $\delta\bar{\mathbf{x}}(t)$ which is the variation in the kinetic part of the action. A simple solution is to revert the Lorentz transformation falling back into the kind of calculation we performed (there might be other options depending on the case). Yet, being just a matter of calculations, the results will not depend on our choices.

Kaufmann's original experiments are from 1900-1904 [Kaufmann, 1906], Lorentz transformations were put in sure ground by Poincaré by 1906 [Poincaré, 1906b] while the Lorentz force and its relation with the action was known since 1892 [Lorentz, 1892]. Thus, early in the XXth Century all the elements to put the puzzle together were available. The failure in so doing must be searched in the mind of the scientists and their new (and in those times tumbling) epistemology where concepts such as consilience had been eradicated while abstraction had been replaced by analogy.

The reasons why Lorentz' action was left behind are not clear. Without it, Lorentz' force becomes an equation herded with Maxwell equations. The action more than the force is the link with Newton's mechanics, without the deduction of the force from the action by the classical variation associated to forces in Hamilton's mechanics, several forms of gluing the force into the mechanics emerge. For exampling, multiplying Newton's equation by $\gamma(u)$ the "relativistic Lagrangian" (see [Goldstein, 1980]) $\mathcal{L} = -m/\gamma(u)$ emerges. However, if this new kinetic energy is used with Hamilton's principle one runs into inconsistencies with the Doppler effect. This is the price paid for instrumentalism: the lost of consilience. We must indicate that it appears impossible to imagine how the force is transmitted. Intuition conceives two ways of transmitting actions from a source to a detector, namely throwing a stone or making waves in a fluid, but none of them will work. The theorem explaining that actions can only occur by analogy of these childish forms has not been proven, it is only a psychological limitation, the limitation Newton got rid of: I frame no hypotheses!

Poincaré discussed psychological aspects of Lorentz' contraction [Poincaré, 1906a, 1913c] in a very insightful way, beginning to show how entrenched in our minds is the idea of absolute space. The following paragraph may convince the reader:

> Suppose that in the night all the dimensions of the universe become a thousand times greater; the world will have remained *similar* to itself, giving to the word *similitude* the same meaning as in Euclid, Book VI. Only what was a meter long will measure thenceforth a kilometre, what was a millimetre long will become a meter. The bed whereon I lie and my body itself will be enlarged in the same proportion. When I awake to-morrow morning, what sensation shall I feel in the presence of such an astounding transformation? I shall perceive nothing at all. The most precise measurements will be incapable of revealing to me anything of this immense convulsion, since the measures I use will have varied precisely in the same proportion as the objects I seek to measure. In reality, this convulsion exists only for those who reason as if space were absolute. If I for a moment have reasoned as they do, it is in order the better to bring out that their way of seeing implies contradiction. In fact it would be better to say that space being relative, nothing at all has happened, which is why we have perceived nothing.

The simple fact that we can follow Poincaré's imagination tell us how easily we think in terms of a fantasy: absolute space. As Poincaré explains, Lorentz would say (think of eq. (4.C.2)) that space has contracted and consistent with it he would have to say that mass depends on velocity. Alternatively, he could have chosen to say that the velocity has increased by a factor γ. In all cases what is being used is the idea that for a body in motion the time it takes to move in front of us, Δt, relates to a known (but not measured) quantity, L in the form

$\Delta t = L/w$ where w will be called the velocity. Unless we can measure the three quantities, $\Delta t, L, w$ we cannot test the quality of the inference. The formula comes from elementary physics and was inferred in the context of motion of bodies at small relative velocities. Hence, its use is acceptable under the condition that the symbols refer to bodies having small velocities relative to the detectors. Can we propose an experiment to measure the three of them? A modification of the famous train paradox of SR will do. The train has length L measured in a system at rest with respect to it. On the side of the train there are two sources of light of small linear dimensions as compared to L: one in the front with a sharply defined green light (measured at rest with respect to the train) and one at the tail with red light. The distance between the centres of these spots is precisely the "length of the train", L. On the platform we have two small detectors that will stay at negligible distance of the train when it passes the station. The detectors send signals when they detect any light. They are placed at a distance L on the station, the one further away from the coming train logs the time when the "green" light is received (Doppler shifted) and the one closer to the incoming train does the same with the "red" light. Finally, our detectors give us the frequency of the light received. If the time in which these signals are registered in both detectors is in coincidence using synchronised clocks (which are not in relative motion so the synchronisation cannot be objected) the length of the train is L since it is a proper measurement of length. The time interval Δt is the time elapsed between detection of green and red light by one and the same detector. Finally, we can measure an electromagnetic velocity using the Doppler shift for the transverse direction, $\frac{\nu'}{\nu} = \gamma$.

Our expectations are: the "green" light is registered by the green sensor simultaneously with the detection of the "red" light by the red sensor, the reason being that the relation of each light to its corresponding sensor is the same, it is the same experiment just shifted a fixed distance. Let us make clear this point. The electromagnetic contact takes place in both cases when time and space differences are zero and this property is invariant under Lorentz transformations, thus it is zero for train and platform, and for the two sets of source-detector pairs, each of them considering itself the centre of reference but pairwise coordinated in time: detectors on the platform and lights on the train.

Also, the time interval Δt is given by eq. (4.C.2) with reference to the mechanical velocity u. The theory developed says then that $\nu' \Delta t = \frac{L}{u}\nu$, this is: the number of oscillations of the EM field perceived during the transit of the train is equal to the number of oscillations produced between both events. The frequency ν can be counted perhaps in a certain number of oscillations of a mechanical pendulum, thus the rhs is mechanical and the lhs electromagnetic. The number of tics of the clocks however remains the same.

As Poincaré pointed out

> If the wave surfaces of light had undergone the same deformations as the material bodies we should never have perceived the Lorentz-Fitzgerald deformation.

Thus, to perceive a deformation something must remain without deformation for comparison and it is then impossible to say whether the body has contracted or the measuring device has elongated, all what we truly know is that some relation has changed. In the present case it would appear as if we can make an arbitrary choice peaking either the gravitational-clock or the EM-clock as unchanged. However, if we select the EM-clock we have to rewrite all of mechanics since it is based upon the contrary choice. Actually, EM itself has been constructed on the basis of the intuited space and time, an intuition that matches the ticking of pendulums. This means that we have to thaw the branch between us and the tree, certainly not a promising form of progress.

If space is a psychological construction based in having ourselves as reference –as Kant, Poincaré, Piaget and this book consider–, then we cannot fail by thinking that there is a space associated to each observer. Pictorially, when there is an EM event, the disturbance produces "jumps" to the space of all the observers to be. But in these jumps it is transformed to a representative according to the relative velocity between observer and the producer of the event. We do not have forms of representing the jump if the relative velocity cannot be considered constant. If it is constant, then the jump is performed by applying a Lorentz transformation to the signal detected by a stationary (with respect to the source) observer. The stationary observer is privileged. It must be taken into account that the producer of the EM event is not just the circuit we are manipulating but the EM environment as well, in theory the universe minus the observer, in practice all material that may influence the observer and the source in a significant amount. The relation between observations is then represented by the Poincaré-Lorentz group. It is important to notice that it is at this point where the analogy with bodies fails. The body is supposed to move carrying its identity with it. But the EM signal changes when it jumps to the space of the observer. In the space of the observer it moves with velocity C, but this is the only property that is preserved. The "photon" that jumps to an observer's space adapts its frequency to the new space, how could it be said to be the same photon if we do not mode out the Poincaré-Lorentz group? Thus, photons cannot represent the EM action except in its mechanical aspects. There is nothing real in such a photon of frequency ν, since the frequency depends on the signal and the observer as well.

4.D. Scholium. The concept of *consilience*

Along this chapter we have shown that what we know of electromagnetism is compatible with the space-time used to construct it; that the delayed action at distance (DAD) can produce, when used in conjunction with Hamilton's least action principle, Maxwell equations, the continuity equation, Lorentz' force, explain Kaufmann's results and explain why Newton's third law does not hold for DAD but it is recovered for instantaneous action at distance. Everything without resorting to ad-hoc hypotheses (be it a dependence of mass with the velocity, an ad-hoc relation between velocity and kinetic energy or others)[55].

[55] It is a common belief that Newton's mechanics is recovered from SR in the limit $C \to \infty$, however, the claim is false. For $C \neq 0$ there is no relation proposed by SR that has as limit the third law. The limit of "no relation" for any $C > 0$ is "no relation". The third law cannot be recovered in current settings of electromagnetism.

W. Whewell coined the word "consilience" [XIII-XVI, Whewell, 1840]

> XIII. *Induction* is a term applied to describe the *process* of a true Colligation of Facts by means of an exact and appropriate Conception. An *Induction* is also employed to denote the *proposition* which results from this process.
>
> XIV. *The Consilience of Inductions* takes place when an Induction, obtained from one class of facts, coincides with an Induction, obtained from another different class. This Consilience is a test of the truth of the Theory in which it occurs.
>
> XV. An Induction is not the mere sum of the Facts which are colligated. The Facts are not only brought together, but seen in a new point of view. A new mental Element is *superinduced*; and a peculiar constitution and discipline of mind are requisite in order to make this Induction.
>
> XVI. Although in Every Induction a new conception is superinduced upon the Facts; yet this once effectually done, the novelty of the conception is overlooked, and the conception is considered as a part of the fact.

From a constructivist point of view, Piaget and García [Piaget and García, 1989] present the same idea under the name of "cognitive surpass" adding that cognitive surpass is a logical/constructive operation performed in the developing of children. Piaget and García identify instances of cognitive surpass in the developing of science as well. Actually, Piaget describes as "disequilibria and re-equilibration" the processes indicated by Whewell. The disequilibria occurs because a new phenomenon challenges our beliefs and the re-equilibration when the ampliative hypothesis has been naturalised after changing our beliefs.

The process of induction with consilience (abduction in Peirce) is an endogenous process proper of reasoning. The adaptation of human beings to their environment depends on it and as such it ranks at the same level that deductive reason. Both are trusted because they have given proven service to all of us. The difference between them is that deductive logic is mechanistic and retroductive inference (abduction) has a dialectical nature. The difficulties with dialectics experienced by many logicians have already been discussed in Subsection 1.B.2.

CHAPTER 5

Reconstructing quantum mechanics from electromagnetism

5.1. Introduction

Background. Under the instrumentalist guide, the period 1870-1930 witnessed the consolidation of Electromagnetic theory (EM) and the birth of quantum mechanics (QM). The success of both theories as predictive tools (and thus as mediators of technological development) can hardly be challenged. The interpretative part is often modelled resorting to mechanical analogies [Mach, 2012, Boltzmann, 1974], although a conflict arose with electromagnetism since a mechanical (Galilean) framework and the constancy of the velocity of light appeared to be incompatible.

The underlying model in Bohr's atomic theory is also mechanical, with a planetary electron revolving around the atomic nucleus. A conflict between theories becomes evident since an accelerated charge is expected to radiate, thus losing energy and collapsing towards the nucleus. The solution to the conflict was to negate it: In the atomic realm radiation is not present and electronic orbits are stable. In Bohr's atomic model, stable orbits have to satisfy a quantisation condition (that the angular momentum is an integer multiple of Planck's constant, $\hbar$) and electrons can only undergo transitions among these stable orbits. Nothing is said in the model about the existence of other states. Later, the Schrödinger model will assume that the possible states of the atom are linear combinations of stationary states and stable as well. In both models all possible atomic states are stable and radiation is suppressed. However, it is not possible to ascertain which is the state of the atom since measurements force the state to "collapse" into a stationary (measurable) energy state. This dogma is often indoctrinated by resource to Schrödinger's *cat paradox*. It would appear that the laws of EM are then different for microscopic and macroscopic distances. The possibility of encompassing EM and QM in one broader theory is therefore hopeless in their currently accepted shape since EM and QM are contradictory in their foundations.

The EM that is acceptable for atomic physics displays a number of phenomena that are very difficult to explain. Consider e.g., spontaneous emission (overwhelmingly verified in Laser physics [Carmichael, 1999, Chow et al., 1994]) or the reduction postulate of QM transforming the state of the system at the moment of measurement. A natural question arises: At what distance is it mandatory to change theories? When and how EM ceases being applicable and QM is a must? Are there other experimentally observable consequences? The theories now rest on elements which are not amenable to experimental examination and moreover, there are questions that cannot be asked.

EM also suffers instrumentalist constraints on its own. The current EM approach is rooted in the heritage of Hertz, being his preferred interpretation that the EM fields are associated to space (the ether for Hertz) and that forces surround matter[56]. Thus matter and EM fields have different ontogenesis. There exists "empty space", populated by fields (and subsequently by springs, strings and branes), along with a space occupied by matter (originally impenetrable). The realm of matter and with it atomic physics is divorced from the space where fields dwell (despite the field conception being rooted in the idea of force per unit charge). The link between both realms is given mostly by the Lorentz force, which appears as an addendum to Maxwell's equations displaying the Poincaré-Lorentz symmetry [Poincaré, 1906b], after repeated drift in the meaning of the symbols. The velocity, that was relational in Maxwell and in experiments, became velocity relative to the ether and later relative to an inertial frame [Assis, 1994]. The last two velocities have never been measured. For example, when analysing experiments concerning Doppler effect, the standard theory refers to a velocity with respect to an inertial system, but the measured velocity is a relative one. We will not dwell in these issues, the impossibility to physically contrast special relativity has been discussed by [Essen, 1971, 1978], while its logical inconsistency when regarded from an epistemology that encompasses the phenomenological moment was shown in Chapter 2, Appendix 2.A.1. Strictly speaking, EM floats in a realm of ideas partially divorced from observations.

The possibility of unifying EM and QM under a conception that cleaves them in two separate entities became only possible by imposing the quantification of imagined entities that populate "empty space". This is the presently accepted approach, although their original proposers finally rejected it and despite their internal inconsistencies that would force the collapse of the underlying mathematical structure [Natiello and Solari, 2015].

Unfortunately, alternative approaches have little or no room in the scientific discussion. If the outcome in formulae of two theories is practically the same; instrumentalism may be inclined to choose the option with (apparently) less complications. Coming back to Newtonian mechanics, if the outcome elaborated under the assumption of absolute space to sustain the theory is the same as (a subset of) that developed under Newton's concept of "True motion" [Newton, 1687, Solari and Natiello, 2021] (see also Section 1.2.2) instrumentalists may find no benefit in presenting the second –more abstract– theory, to the price of accepting a conflicting concept that cannot be demonstrated[57]. We promptly recognise that

[56] The expression is rather mysterious, but it can be found also in Faraday (closer to reject it than to accept it) and in Maxwell (closer to accept it than to reject it).

[57] Along the same lines, textbooks do not hesitate in presenting a simple but incorrect derivation of the spin-orbit interaction along with an ad-hoc correction called Thomas precession, instead of using the approach that they consider "correct" [Kastberg, 2020, p. 60].

in the academic formation of technical professionals the difference is usually not very problematic, while in the formation of experimental philosophers (or natural philosophers), to decline the historical cognitive truth for instrumentalist reasons appears as completely inappropriate and a safe source of future conflicts[58].

Instrumentalism is a weakened version of the phenomenological conception presented in [Solari and Natiello, 2022a]. This latter approach is rooted in the work of C. Peirce [Peirce, 1994], W. Whewell [Whewell, 1840] and J. Goethe [Goethe, 2009] developed under the influence of rational humanism and nurtured by ideas of I. Kant [Kant, 1787] and G. Hegel [Hegel, 2001a].

The ideas that allow us to traverse alternative paths were presented a long time ago but their development was abandoned half way through. Faraday observed that the more familiar ideas that privilege the concept of matter before the concept of action could be confronted with the approach that recognises that only actions can be detected while matter is subsequently inferred, and as such exposed to the errors belonging in conjectured entities. In this sense, Faraday recognises that matter can only be conceived in relation to action and what we call matter is the place where action is more intense. See the citations from Faraday in Section 4.1. Such matter extends to infinity and is interpenetrable. Faraday's vibrating rays theory eliminates the need to conceive an ether. The relational electromagnetism initially developed by W. Weber [Assis, 1994] under Gauss' influence, was later developed by the Göttingen school and by L. Lorenz [Lorenz, 1867] introducing Gauss' concept of delayed action at distance. Maxwell expressed his surprise when a theory that was based in principles very different from his own could provide a body of equations in such a high accordance [Maxwell, 2003]. From the Hertzian point of view both theories are basically "the same". However, when experiments later demonstrated the absence (nonexistence) of a material ether [Michelson and Morley, 1887] the rational approach of Faraday-Gauss-Lorenz was already forgotten, probably because the epistemological basis of science had moved towards instrumentalism.

The development of quantum mechanics continued from Bohr's model into Heisenberg's matrix computations, a completely instrumental approach devoid of a priori meaning [Born, 1955]. While the ideas of de Broglie [de Broglie, 1923, 1924] and Schrödinger [Schrödinger, 1926] revolutionised the field, at the end of the Fifth Solvay conference (1927) the statistical interpretation of quantum theory collected most of the support [Schrödinger, 1995]. It is said that only Einstein and Schrödinger were not satisfied with it. Einstein foresaw conflicts with relativity theory, later made explicit in the Einstein-Podolsky-Rosen paper [Einstein et al., 1935]. Schrödinger, after several years of meditation and several changes of ideas, came to the conclusion that the statistical interpretation was forced by the not properly justified assumption of electrons and quantum particles being represented by mathematical points [Schrödinger, 1995]. Certainly, Schrödinger knew well that no statistical perspective was introduced in his construction of undulatory mechanics. Actually, the statistical interpretation of quantum mechanics follows

[58] The distinction between the formation of professionals and philosophers is in [Kant, 1798].

strictly Hertz' philosophical attitude of accepting the formulae but feeling free to produce some argument that makes them palatable. We have called this approach "the student's attitude" [Solari et al., 2024] since it replicates the attitude of the student that is forced to accept the course perspective and has to find a form to legitimise the mandate of the authority: the result is given and the justification (at the level of plausibility) has to be provided.

The standard QM then has to find a meaning for its formulae and such task has been called "interpretation of QM". All the interpretations that we know about rest on the idea of point particles. This is an idea that allows for the immediate creation of classical images that suggest new formulae such as those entering in the atomic spectra of the Hydrogen atom. Since the accelerated electron (and proton) cannot radiate in QM by ukase[59], the radiation rule has to be set as a separate rule. The atom, according to the standard theory, can then be in an "mixed energy" state until it is measured (meaning to put it in contact with the laboratory), what makes the state represented by the wave function to collapse (in the simplest versions). While there are more sophisticated[60] interpretations, none of them incorporates the measurement process into QM: It remains outside QM.

The idea of isolating an atom deserves examination. One can regard as possible to set up a region of space in which EM influences from outside the region can be compensated to produce an effective zero influence. In the simplest form we think of a Faraday cage and perhaps similar cages shielding magnetic fields. In the limit, idealised following Galileo, we have an atom isolated from external influences. But this procedure is only one half of what is needed. We need the atom to be unable to influence the laboratory, including the Faraday cage. And we cannot do anything onto the atom because so doing would imply it is not isolated. This is, we must rest upon the voluntary cooperation of the atom to have it isolated. The notion of an isolated atom is thus shown to be a fantasy. If the atom, as the EM system that it is, is in the condition of producing radiation, it will produce it whether the Faraday cage is in place or not. The condition imposed by the ukase is metaphysical (i.e., not physical, nor the limit of physical situations). Actually, if such condition is imposed to Hamilton's principle, Schrödinger's time-evolution equation is recovered to the price that the consilience with electromagnetism and classical mechanics is lost.

It was Schrödinger who dug more deeply into the epistemological problems of QM. His cat, now routinely killed in every QM course/book was a form of ridiculing the, socially accepted, statistical interpretation of QM:

> One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays,

[59] ukase: edict of the tsar.

[60] We imply the double meaning as elaborated and false. The first as current use and the second implied in the etymological root "sophism" in the ancient Greek: σοφιστικός (sophistikós), latin: sofisticus.

> but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The psi-function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts.[Schrödinger, 1980]

Schrödinger understood as well that the need for a statistical interpretation was rooted in the assumption of point particles [Schrödinger, 1995].

In the present work, the conservation of energy of a quantum particle is described in the form in which Faraday conceived matter as the inferred part of the matter-action pair and it leads directly to the transition rule: $\mathcal{E}_I^f + \mathcal{E}_{EM}^f = \mathcal{E}_I^i + \mathcal{E}_{EM}^i$ with i, f meaning initial and final states while I, EM mean internal and EM energies. In the early times of QM, Bohr showed that the Balmer and Rydberg lines of the Hydrogen atom corresponded to this rule. This is to say that the observed values summarised in Rydberg's formula agree in value with the equation, yet the equation was not part of the theory because the "emission" of light was conceived as the result of external influences since the isolated atom could not decay and in such way it would not collapse with the electron falling onto the proton.

5.2. Abduction of quantum (wave) mechanics

We begin by highlighting how QM enters the scene as a requirement of consistence with electromagnetism. Consider a point charge, q, moving at constant velocity v with respect to a system that acts as a reference. We associate with this charge the relational charge-current density

$$\begin{pmatrix} Cq \\ qv \end{pmatrix} \delta(x - vt) = \begin{pmatrix} C\rho \\ j \end{pmatrix} (x, t).$$

The association of current with *charge-times-velocity* goes back to Lorentz, Weber and possibly further back in time and it is valid at least for low velocities, where $|\frac{v}{C}| \ll 1$ (see Section 4.C).

The velocity can then be written as the quotient between current amplitude and charge, $\beta = \dfrac{qv}{Cq}$. At the same time, Lagrangian mechanics associates the velocity with the momentum, p, of a free moving body by a relation of the form, $\beta = \dfrac{p}{Cm}$. We propose that charge and mass are constitutive attributes of microscopic systems and hence

$$\begin{pmatrix} mC \\ p \end{pmatrix} = \frac{m}{q} \begin{pmatrix} Cq \\ qv \end{pmatrix}$$

will behave as the charge-current density when intervening in electromagnetic relations.

If rather than a point particle we accept an extended body moving at constant velocity without deforming and represented by a charge-current density

$$\begin{pmatrix} Cq \\ qv \end{pmatrix} \rho(x,t)$$

where $\int d^3x\,\rho(x,t) = 1$, we find that the conditions posed above can be satisfied whenever

$$\rho(x,t) \propto \int d^3k\,\phi(k)\exp(i(k\cdot x - wt))$$

with $w = f(k)$ (to be addressed below). If we further associate the frequency with the (kinetic) energy in the form $E = \hbar w$, as de Broglie did [de Broglie, 1924], and write $p = \hbar k$ we have

$$E \;=\; \hbar f(\frac{p}{\hbar}).$$

Since we expect $\rho(x,t)$ to be real, it is clear that this association is not enough to rule out the unobservable negative energies, since
$\rho = \rho^* \Leftrightarrow \phi(k)\exp(i(k\cdot x - \frac{E}{\hbar}t)) = \phi^*(k)\exp(i((-k)\cdot x - (-\frac{E}{\hbar})t))$. Negative energies would be required and they would rest on an equal footing as positive ones. Introducing the quantum-mechanical wave function ψ satisfying

$$\begin{aligned}\psi(x,t) &= \left(\frac{1}{2\pi}\right)^{3/2}\int d^3k\,\phi^*(k)\exp(\frac{i}{\hbar}(p\cdot x - Et)) \\ &= \left(\frac{1}{2\pi\hbar}\right)^{3/2}\int d^3p\,\phi^*(\frac{p}{\hbar})\exp(\frac{i}{\hbar}(p\cdot x - Et)) \qquad (5.2.1) \\ \rho(x,t) &= q\;\psi(x,t)^*\psi(x,t)\end{aligned}$$

we automatically obtain a real charge-density without the imposition of negative energies (where for free particles $E \equiv T$, the kinetic energy, whose expression will be derived below).

We retain from traditional quantum mechanics the idea of expressing physical observables as real valued quantities associated to an *operator* integrated over the whole entity.

We recall that the quantum entity is a unity and *not* a mathematical point. Hence, objects like $(\psi^*\,(-i\hbar\nabla\psi))$ do not necessarily have physical (measurable) meaning. Further, observables depending only on x are multiplicative operators in front of $\psi(x,t)$. For the case of the position operator x we have[61],

$$\begin{aligned}\langle\hat{x}\rangle &= \int d^3x\;\psi^*(x,t)\;x\;\psi(x,t) \\ &= \int d^3x\;d^3p\;d^3p'\;\frac{1}{(2\pi\hbar)^3}\Phi^*(p')\exp(\frac{i}{\hbar}(p'\cdot x - Et))\;x\;\Phi(p)\exp(-\frac{i}{\hbar}(p\cdot x - Et)) \\ &= \int d^3p\;d^3p'\;\Phi^*(p')\;\;(i\hbar\nabla_p\;\Phi(p))\int d^3x\;\frac{1}{(2\pi\hbar)^3}\exp(\frac{i}{\hbar}(p-p')\cdot x) \\ &= \int d^3p\;\Phi^*(p)\;(i\hbar\nabla_p\Phi(p))\end{aligned}$$

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[61]Integrals such as $\langle\psi^* x\psi\rangle = \int d^3x\,(\psi^* x\psi)$ or $\langle\psi^*(-i\hbar\nabla\psi)\rangle = \int d^3x\,(\psi^*(-i\hbar\nabla\psi))$ have been called *expectation values* in the traditional probability interpretation of quantum mechanics.

while[62]

$$\hat{E} \quad = \quad i\hbar\partial_t.$$

De Broglie's proposal and eq. (5.2.1) relate x and p through a Fourier transform and hence the momentum operator becomes $\hat{p} = m\hat{v} = -i\hbar\nabla$. Thus, in the case of a free charged particle we arrive to Schrödinger's equation

$$i\hbar\partial_t\psi = \hat{E}\psi. \tag{5.2.2}$$

We have yet to find $f(k)$. If we consider that eq. (5.2.2) is the definition of the Hamiltonian operator $\hat{H} \equiv \hat{E}$, and recall that the operator associated to the current density is $q\hat{v} = \frac{q}{m}\hat{p}$; when in addition the conservation of charge, also known as the *continuity equation*, $\partial_t\rho + \nabla\cdot j = 0$, is considered we obtain the relations

$$\begin{aligned} q\partial_t\rho &= q\left[\psi^*\partial_t\psi + \partial_t\psi^*\psi\right] = \frac{q}{i\hbar}\left[\psi^*(\hat{H}\psi) - (\hat{H}\psi)^*\psi\right] \\ &= -\nabla\cdot\frac{1}{2}\frac{q}{m}\left[\psi^*\hat{p}\psi + (\hat{p}\psi)^*\psi\right] \end{aligned}$$

and expanding the r.h.s. we can see that the only solution is:

$$\begin{aligned} \hat{p} &= -i\hbar\nabla \\ \hat{H} &= \frac{\hat{p}^2}{2m} + U(x) = -\frac{\hbar^2}{2m}\Delta + U(x) \\ j &= \frac{q}{2m}\left(\psi^*\left(-i\hbar\nabla\psi\right) + \left(-i\hbar\nabla\psi\right)^*\psi\right) = -\frac{i\hbar q}{2m}\left(\psi^*\left(\nabla\psi\right) - \left(\nabla\psi\right)^*\psi\right) \end{aligned}$$

where $U(x)$ is any scalar function of the position. We adopt $U(x) = 0$ as for a free particle there is no justification for a potential energy, hence $\hbar f(k) = \frac{\hbar^2}{2m}|k|^2$. We have then found the free particle Hamiltonian consistent with de Broglie's approach and the assumption $\rho(x,t) = |\psi(x,t)|^2$.[63]

In summary, QM results from requirements of compatibility with electromagnetism following an abduction process in which the hypothesis $E = \hbar w$ results as an extension of the photoelectric relation. As a consequence, any attempt to superimpose further formal relations with electromagnetism such as requiring invariance or equivariance of forms under the Poincaré-Lorentz group is bound to introduce more problems than solutions.

5.2.1. Expression of the electromagnetic potentials.

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[62]Throughout this work the notations $\partial_t\psi$, $\dot{\psi}$ and $\psi_{,t}$ all denote the same concept, namely partial derivative with respect to time.

[63]In his original work, de Broglie [de Broglie, 1924] assumed the relation, $E^2 = (m^2C^2 + p^2)C^2$ proposed by special relativity on an utilitarian basis as already commented. It was later Schrödinger [Schrödinger, 1926] who proposed the classical form based on the correspondence with Newton's mechanics.

5.2.1.1. *Physical Background.* In this section our goal is to develop an electromagnetic theory of microscopic systems encompassing both QM and Maxwell's electrodynamics. Hence, we will compute Maxwell's EM interaction energy, either between the microscopic system and external EM-fields or between different fields arising within the system. We display first some classical results that are relevant for the coming computations.

Maxwell's electromagnetic *interaction energy* between entities 1 and 2 reads:

$$\mathcal{E} = \int d^3x \left[\epsilon_0 E_1 \cdot E_2 + \frac{1}{\mu_0} B_1 \cdot B_2 \right]. \tag{5.2.3}$$

The classical EM potentials originating in a charge density $|\psi|^2$ and a current $-\frac{i\hbar q}{2m}\left(\psi^*(\nabla_y\psi) - (\nabla_y\psi)\psi\right)(y,t)$, as defined in the Lorenz gauge, read

$$V(x,t) \quad = \quad \frac{q}{4\pi\epsilon_0} \int d^3y \; \frac{1}{|x-y|} \left|\psi(y, t - |x-y|/C)\right|^2$$

for the scalar potential (where as usual $C^2 = (\mu_0\epsilon_0)^{-1}$), while the vector potential is:

$$A(x,t) \quad = -\frac{i\hbar}{2} \quad \frac{\mu_0 q}{4\pi m} \int d^3y \; \frac{1}{|x-y|} \left(\psi^*(\nabla_y\psi) - (\nabla_y\psi)\psi\right)(y, t - |x-y|/C).$$

Finally, electric and magnetic fields are obtained through Maxwell's equations $E = -\nabla V - \dfrac{\partial A}{\partial t}$ and $B = \nabla \times A$. We recall from Maxwell (see e.g. [[630] Maxwell, 1873]) that all electromagnetic quantities and the wave function are assumed to vanish sufficiently fast at infinity. This allows for the use of partial integrations and Gauss' theorem whenever appropriate.

5.2.1.2. *Action integral and the least action principle.* Lorentz's electromagnetic action,

$$\mathcal{A} = \frac{1}{2} \int dt \int \left(\frac{1}{\mu_0}|B|^2 - \epsilon_0 |E|^2 \right) d^3x$$

(see eq. (4.3.6)) accounts for the propagation of electromagnetic effects but not for the sources. In contrast Schrödinger's equation for the free charge can be derived from a variational principle

$$\mathcal{A}_{QM} = \int_{t_0}^{t_1} ds \left\langle \frac{1}{2}\left(\psi^* \left(i\hbar\dot{\psi} \right) + (i\hbar\dot{\psi})^*\psi \right) + \frac{\hbar^2}{2m}\psi^*\Delta\psi \right\rangle$$

where $\langle\rangle$ indicates integration in all space and Δ is Laplace's operator. The Schrödinger equation is recovered when the QM-action is varied with respect to ψ (or ψ^*):

Lemma 5.1. *$\mathcal{A}_{QM}$ is stationary with respect to variations of ψ^* such that $\delta\psi^*(t_0) = 0 = \delta\psi^*(t_1)$ if and only if $i\hbar\dot{\psi} = -\dfrac{\hbar^2}{2m}\Delta\psi$.*

We display the proof in Appendix 5.A.

The hope is then that by adding both actions, $\mathcal{A}_{EM}$ that almost ignores the dynamics of the sources and $\mathcal{A}_{QM}$ that ignores the associated fields we can propose a synthetic action integral

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_{EM}+\mathcal{A}_{QM}\\ &= \int_{t_0}^{t_1} ds\left\langle \frac{1}{2}\left(\psi^*\left(i\hbar\dot{\psi}\right)+(i\hbar\dot{\psi})^*\psi\right)+\frac{\hbar^2}{2m}\psi^*\Delta\psi+\frac{1}{2}\left(\frac{1}{\mu_0}B^2-\epsilon_0E^2\right)\right\rangle \end{aligned} \quad (5.2.4)$$

no longer separating effects from sources, since, as stated in subsection 5.2, in the present construction all of: current, charge-density and electromagnetic potentials depend of the wave function ψ. Hamilton's variational principle applied to the action integral $\mathcal{A}$ yields all the dynamical equations in EM, and QM.

5.2.2. Variations of the action. The variations we are going to consider will be the product of an infinitesimal real function of time denoted ϵ when it is a scalar and $\bar{\epsilon}$ (a vector of arbitrary constant direction, $\bar{\epsilon}=\epsilon k, \epsilon r, \epsilon\Omega$) when it refers to a wave number, position or solid-angle respectively; times a function μ often of the form $\mu=\hat{O}\psi$ for some appropriate operator $\hat{O}$. The variation of the wave function is reflected directly in variations on the charge-density, current, and fields. We will use for them the symbols $\delta\rho, \delta j, \delta E, \delta B$ for a set of variations that satisfy Maxwell's equations.

Hence, according to the previous paragraph, the variation $\delta\psi=\epsilon\mu$ is reflected in a variation of the EM quantities as follows:

(1) Variation of charge-density: $\delta\rho=\epsilon q\left(\mu^*\psi+\psi^*\mu\right)\equiv\epsilon\mathfrak{X}$.
(2) Variation of current: $\delta j=\epsilon\mathfrak{J}_1+\dot{\epsilon}\mathfrak{J}_2$. We will discuss in due time the connection between $\delta\psi$ and δj.

The continuity equation then reads

$$(\delta\rho)_{,t}+\nabla\cdot\delta j=0,$$

which discloses into

$$\begin{aligned}\mathfrak{X}+\nabla\cdot\mathfrak{J}_2 &= 0\\ \mathfrak{X}_{,t}+\nabla\cdot\mathfrak{J}_1 &= 0\end{aligned}$$

thus demanding $\nabla\cdot\mathfrak{J}_1-\nabla\cdot\mathfrak{J}_{2,t}=0$ and suggesting $\mathfrak{J}_2=-\epsilon_0\mathfrak{E}$, with $\epsilon_0\nabla\cdot\mathfrak{E}=\mathfrak{X}$ for some electrical field that needs yet further specification.

We develop here the mathematical results of this Chapter. Proofs can be found in Appendix 5.A.

Lemma 5.2. *(Preparatory lemma). Under the previous definitions and the conditions* $\mathfrak{X}+\nabla\cdot\mathfrak{J}_2=0$ *and* $\left(\mathfrak{X}\right)_{,t}+\nabla\cdot\mathfrak{J}_1=0$, *complying with the continuity equation, a variation* $\delta\psi=\epsilon\mu$ *of the wave function produces the varied action integral*

$$\begin{aligned}\delta\mathcal{A} &= \delta\int_{t_0}^{t_1} ds\left\{\left\langle\frac{1}{2}\left(\psi^*\left(i\hbar\dot{\psi}\right)+(i\hbar\dot{\psi})^*\psi\right)+\frac{\hbar^2}{2m}\psi^*\Delta\psi\right\rangle+\frac{1}{2}\left\langle\frac{1}{\mu_0}B^2-\epsilon_0E^2\right\rangle\right\}\\ &= \int_{t_0}^{t_1}\epsilon\, ds\left\langle\left(\mu^*\left(i\hbar\dot{\psi}\right)+(i\hbar\dot{\psi})^*\mu\right)+\frac{\hbar^2}{2m}\left(\mu^*\Delta\psi+\mu\Delta\psi^*\right)\right\rangle+\\ &\quad+\int_{t_0}^{t_1}\epsilon\, ds\left\langle A\cdot\mathfrak{J}_1-\left(A\cdot\mathfrak{J}_2\right)_{,t}-V\mathfrak{X}\right\rangle,\end{aligned}$$

Variation	$\delta\psi$	$\mathfrak{X}$	$\mathfrak{J}_1$	$\mathfrak{J}_2$
time	$\epsilon\partial_t\psi$	$\rho_{,t}$	$-2\epsilon_0\mathfrak{L}_{,t} - \mathfrak{j} = j_{,t}$	j
position	$(i\epsilon k \cdot x)\,\psi$	0	0	0
displacement	$\epsilon r \cdot \nabla\psi$	$r \cdot \nabla\rho$	$(r \cdot \nabla)\, j = \mathfrak{j}$	$-r\rho$
rotation	$\epsilon\Omega \cdot (x \times \nabla\psi)$	$\Omega \cdot (x \times \nabla\rho)$	$(\Omega \cdot (x \times \nabla))\, j - \Omega \times j = \mathfrak{j}$	$-\Omega \times (x\rho)$

TABLE 5.2.1. Variations

Variation/result	$\delta\psi$	*Equation of motion*
time/energy	$\epsilon\partial_t\psi$	$\partial_t \left\langle -\frac{\hbar^2}{2m}\psi^*\Delta\psi + \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\right\rangle = 0$
position/velocity	$(i\epsilon k \cdot x)\,\psi$	$\partial_t \langle\psi^* x\psi\rangle = \frac{\langle j\rangle}{q}$
displacement/momentum	$\epsilon r \cdot \nabla\psi$	$\frac{m}{q}\partial_t \langle j\rangle = \langle j \times B + \rho E\rangle$
rotation/angular momentum	$\epsilon\Omega \cdot (x \times \nabla\psi)$	$\frac{m}{q}\partial_t \langle x \times j\rangle = \langle x \times (\rho E + j \times B)\rangle$

TABLE 5.2.2. Equations of motion.

where the fields are denoted as $B = B_m + B_r$ and $E = E_m + E_r$, with the index m corresponding to the microscopic system to be varied and r denotes the "outside world". A, V are the electromagnetic potentials.

COROLLARY 5.1. *Without loss of generality, we set $\mathfrak{J}_2 = -\epsilon_0\mathfrak{L}$, where $\epsilon_0\nabla \cdot \mathfrak{L} = \mathfrak{X}$.*

We will address different relations in a few related results which are stated separately because of expository convenience.

THEOREM 5.1. *(Classical variations theorem). Under the assumptions of Lemma 5.2, the null variation of the action*

$$\mathcal{A} = \int_{t_1}^{t_2} ds \left(\left\langle \frac{1}{2}\left(\psi^*\left(i\hbar\dot{\psi}\right) + (i\hbar\dot{\psi})^*\psi\right) + \frac{\hbar^2}{2m}\psi^*\Delta\psi\right\rangle + \frac{1}{2}\left\langle\frac{1}{\mu_0}B^2 - \epsilon_0 E^2\right\rangle\right)$$

as a consequence of a variation of the wave function as:

$$\delta\psi = \epsilon\psi_{,t},\ (i\epsilon k \cdot x)\,\psi,\ (\epsilon\bar{x} \cdot \nabla\psi)\,,\ (\epsilon\Omega \cdot (x \times \nabla\psi))$$

–as summarised in Table 5.2.1– results in the corresponding equations of motion displayed in Table 5.2.2.

The entries in table 5.2.1 were produced following intuitions based upon the notion of conjugated variables in classical mechanics. A displacement in time is associated with the law of conservation of energy, one in space with the law of variation of linear momentum and an angular displacement with angular momentum. The variation leading to the derivative of the position is of an abductive nature since what it is really known is the kinematic relation. Hence, the variation is obtained as the answer to the question: which variation produces the kinematic relation $\langle x\rangle_{,t} = \langle j\rangle / q$?

Variation	$\delta\psi$	$\mathfrak{X}$	$\mathfrak{j}$	$\epsilon_0\mathfrak{E}$	$\nabla\times\mathfrak{B}$
time	$\epsilon\partial_t\psi$	$\rho_{,t}$	$j_{,t}$	$E_{,t}$	$\nabla\times B_{,t}$
position	$(i\epsilon k\cdot x)\,\psi$	0	0	0	0
displacement	$\epsilon r\cdot\nabla\psi$	$(r\cdot\nabla)\,\rho$	$(r\cdot\nabla)\,j$	$r\rho$	$-\nabla\times(r\times j)$
rotation	$\epsilon\Omega\cdot(x\times\nabla\psi)$	$\Omega\cdot(x\times\nabla\rho)$	$(\Omega\cdot(x\times\nabla))\,j-\Omega\times j$	$\Omega\times(x\rho)$	$-\nabla\times((\Omega\times x)\times j)$

TABLE 5.2.3. Variations in Hamiltonian form

The proposed variations do not follow a general pattern. Hence, we expect that there might be alternative forms of variations that lead to identical results which in addition present consilience. Moreover, we expect that the energy would take the place of the Lagrangian since this is one of the intuitions based upon experimental observations that lead to quantum mechanics. We present the result of this synthetic steps in the next theorem.

There are four matters that have to be put together to complete the construction of quantum mechanics:

(1) Changing the Lagrangian presentation into a Hamiltonian one.
(2) Showing that the Hamiltonian presentation has consilience and produces the particular results of Theorem 5.1 as well (it is a generalisation/correction of the particular cases shown).
(3) Showing that this setting produces the usual eigenvalue-eigenvector equations of quantum mechanics for the stationary wave functions.
(4) Making more explicit the relation between the variation of the wave function and the variation of the electromagnetic current.

We state these results as three Lemmas and a Corollary.

LEMMA 5.3. *(Hamiltonian form) Let the set of functions* $\mathfrak{X},\mathfrak{j},\mathfrak{E},\mathfrak{B}$ *satisfy Maxwell equations with* $\delta\rho=\epsilon\mathfrak{X}=\epsilon q\,(\mu^*\psi+\psi^*\mu)$ *as in Lemma 5.2 . If we let the current be* $\delta j=\epsilon\mathfrak{I}_1+\dot{\epsilon}\mathfrak{I}_2$*, with*

$$\begin{aligned}\mathfrak{I}_2 &= -\epsilon_0\mathfrak{E}\\ \mathfrak{I}_1 &= -2\epsilon_0\mathfrak{E}_{,t}-\mathfrak{j}\end{aligned}$$

the null variation of the action integral satisfies

$$\left\langle\mu^*\left(i\hbar\dot{\psi}\right)+\left(i\hbar\dot{\psi}\right)^*\mu\right\rangle+\frac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi+\mu\Delta\psi^*\right\rangle-\left\langle\frac{1}{\mu_0}B\cdot\mathfrak{B}+\epsilon_0E\cdot\mathfrak{E}\right\rangle=0, \tag{5.2.5}$$

where $-\frac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi+\mu\Delta\psi^*\right\rangle+\left\langle\frac{1}{\mu_0}B\cdot\mathfrak{B}+\epsilon_0E\cdot\mathfrak{E}\right\rangle\equiv\mathcal{H}$ *is recognised as the Hamiltonian of the problem.*

COROLLARY 5.2. *The form of the variations appropriated to the Hamiltonian form corresponding to the results of Theorem 5.1 (Table 5.2.2) is the one given in Table 5.2.3.*

LEMMA 5.4. *(Variation of a stationary state). In the special case when* ψ *corresponds to a stationary state,* $\psi(x,t)=\exp(-i\omega t)\,\psi(x,0)\equiv\exp(-i\omega t)\,\psi_\omega$

and assuming that the external electromagnetic fields are time-independent, and B is homogeneous, the variation in Lemma 5.3 yields

$$\langle \mu^* (\hbar\omega\psi_\omega)\rangle + \frac{\hbar^2}{2m}\langle \mu^* \Delta\psi_\omega\rangle - \left\langle \mu^* \hat{H}_{EM}\psi_\omega\right\rangle = 0 \tag{5.2.6}$$

where

$$\hat{H}_{EM}\psi = V\psi - \left(\frac{q}{2m}\right)(B\cdot(x\times(-i\hbar\nabla))\psi)$$

eq. (5.2.6) *gives the Schrödinger equation for stationary states.*

The original expression by Schrödinger was developed setting $B = 0$ and assuming that the electron is the quantum entity while the proton is an "external" point charge, responsible for the electrostatic potential V.

In Theorem 5.1 as well as in Corollary 5.2 we rested on phenomenological ideas (synthetic judgments) to produce the current associated to the variation. To find a general form for a set of intuitions appears as almost impossible. We proposed in the case of rotations $\mathfrak{j} = (\Omega\cdot(x\times\nabla))\, j - \Omega\times j$ because the rotation of vectors demands to change the spatial argument as well as the orientation of the components. The confirmation of the abductive hypothesis came from the result, which has an intuitively appealing form. On the contrary, had we forgotten the term $-\Omega\times j$ we would have ended up with a result with no consilience with that of classical mechanics. We stress that these are rational arguments which are not of a deductive nature. Nevertheless, there are some mathematical relations that have to be satisfied since the continuity equation must be enforced.

LEMMA 5.5. *The variations $\hat{O}\psi$ (with $\hat{O}_{,t} = 0$) that may produce a varied current $\mathfrak{j}$ fulfilling the conditions of Lemma 5.2 extend as real operators Θ acting on forms such as $(\rho,\ j,\ \nabla\cdot j)$ that can be regarded as real-valued (scalar-or vector) functions of (ψ^*, ψ) as:*

$$\begin{aligned}
\Theta\rho\left(\psi^*,\psi\right) &= \rho\left(\left(\hat{O}\psi\right)^*,\psi\right) + \rho\left(\psi^*,\left(\hat{O}\psi\right)\right)\\
\Theta j\left(\psi^*,\psi,\nabla\psi^*,\nabla\psi\right) &= j\left(\left(\hat{O}\psi\right)^*,\psi,\left(\hat{O}\nabla\psi\right)^*,\nabla\psi\right) + j\left(\psi^*,\hat{O}\psi,\nabla\psi^*,\left(\hat{O}\nabla\psi\right)\right)\\
\Theta\left(\nabla\cdot j\left(\psi^*,\psi,\Delta\psi^*,\Delta\psi\right)\right) &= j\left(\left(\hat{O}\psi\right)^*,\psi,\left(\hat{O}\Delta\psi\right)^*,\Delta\psi\right) + j\left(\psi^*,\left(\hat{O}\psi\right),\Delta\psi^*,\left(\hat{O}\Delta\psi\right)\right).
\end{aligned}$$

Then, Θ preserves the continuity equation as $\Theta\left(\rho_{,t} + \nabla\cdot j\right) = 0$ and $\Theta(\nabla\cdot j) = \nabla\cdot\mathfrak{j}$ when the variation satisfies the relation

$$\psi^*\left(\left[\Delta,\hat{O}\right]\psi\right) = \psi^*\left(\left[\Delta,\hat{O}\right]^\dagger\psi\right) + \nabla\cdot K$$

(where $\dagger$ indicates Hermitian conjugation) for some vector K satisfying $\int d^3x\,(\nabla\cdot K) = \int_\Sigma K\cdot\widehat{d\Sigma} = 0$. In such a case, $\mathfrak{j} = \Theta j + K$. The operator $\hat{O}$ is then anti-Hermitian.

As examples for the application of the previous lemma we use the cases listed in table 5.2.3. The operators associated to energy, velocity and rotation are $\hat{O} = (\partial_t, (r\cdot\nabla), \Omega\cdot(x\times\nabla))$. All of them commute with Laplace's operator, hence $\mathfrak{j} = \Theta j$. In the case of the position $\Theta j = 0$, $\Theta(\nabla\cdot j) = 0$ and the terms including $\left[\nabla,\hat{O}\right]$ add up to zero.

5.2.3. Discussion. In Subsection 5.2.2 we have shown that when the electromagnetic action outside the locus of masses, charges and currents as it appears in the Lorenz-Lorentz theorem is complemented with the action where mass, charge and currents are not necessarily null, the action proposed encodes Maxwell's electromagnetic laws, Lorentz' force and Newton's equations of motion, as well as an extension of Schrödinger equation for stationary states. Thus, electromagnetism does not question the validity of classical mechanics, nor quantum mechanics questions the validity of electromagnetism or classical mechanics. They coexist in a synthetic variational equation.

The difference between Schrodinger's eigenvalue-eigenvector equation and eq. (5.2.6) corresponds to the "static energy" which is present in Maxwell's formulation of electromagnetism [[680] p. 270 Maxwell, 1873]. The method of construction used by Maxwell is not consistent with the notion of quanta since it assumes that any charge can be divided in infinitesimal amounts. The electrostatic energy corresponds then to assembling together the total charge. From a constructive point of view, such energy is not justified.

There are several possible alternatives to Maxwell's method. One of them is not to consider self-action (second alternative). A third alternative is to consider only the energy contained in transversal (rotational) non stationary fields $(E_\perp, B)$ (where $E_\perp = E - (-\nabla V_C)$, being V_C the Coulomb potential). In the first and third cases, there will be a flow of energy associated to the EM waves each time the entity is not in a stationary state. In the second case, energy will correspond to interactions and the possibility of a quantum entity to be isolated as the limit of constructively factual situations will be doubted. One way or the other each view can be incorporated to the formalism, which in turn will provide consistency requirements that can be contrasted against experimental situations. Which alternative(s) must be discarded is to be decided experimentally. The list of alternatives will remain open until the occasion. We will see that the alternatives that present "no self actions in stationary situations" (second and third alternative) account for the energy levels of the hydrogen atom. Yet, if we want to account for the photoelectric effect or the Compton effect we need to have room for incoming or outgoing energy and momentum associated to the EM fields. This makes the third alternative to look more promising.

It is worth noticing that in the statistical interpretation of quantum mechanics, the energy of the field is not considered (no reasons offered) avoiding in such way dealing with an infinite energy that corresponds to the electrostatic (self) energy of a point charge. However, self energies are not ruled out completely since they are needed (in the standard argumentation) to account for the so called spin-orbit interaction. As usual in the instrumentalist attitude, consistence is not a matter of concern.

5.3. Constructive ideas for Quantum Mechanics from Electromagnetism

We borrow some basic ingredients from Maxwell's electrodynamics and from traditional quantum mechanics. These elements will be combined in dynamical equations in the next Section.

(1) The atom is the microscopic entity susceptible of measurements. Proton and electron within the atom do not have a sharp separable identity, they are inferences (or "shadows"). The identity of the atom consists of charge, current, magnetisation(s) and masses.
(2) The experiments of Uhlenbeck & Goudsmidt and Zeeman as well as that of Stern and Gerlach suggest the existence of intrinsic magnetisations. We introduce it following the standard form relying in Pauli's seminal work [Pauli Jr, 1927].
(3) The interactions within the atom and of the atom with detectors are electromagnetic.
(4) Elementary entities such as the proton and the electron have no self interactions. There is no evidence regarding the existence of self energies (constitutive energies) at this level of description, correspondingly, they will not be included.
(5) The experimental detection of properties belonging to the shadows within the atom depends on the actual possibility of coupling them with measurement devices (usually based upon electromagnetic properties). We will stay on the safe side and will not assume as measurable things like the distance from proton to electron or the probability distribution of the relative position between proton and electron. Such "measurements" remain in the domain of fantasy unless the actual procedure is offered for examination. In this respect we severely depart from textbook expositions of quantum mechanics.
(6) The atom as an EM-entity is described with wave functions. Any quantity associated to an atom, such as its EM fields, are represented by operators on the wave function's space.

5.3.1. Expression of the electromagnetic potentials as integrated values on operators.

5.3.1.1. *Wave functions.* Two bodies that are spatially separated are idealised as represented by a *wave function* which is the product of independent wave functions on different coordinates. For the case of a proton and an electron we write $\Psi(x_e, x_p) = \psi_e(x_e)\psi_p(x_p)$ and call it the *electromagnetic limit.* In contrast, a single body with an internal structure is represented by the product of a wave function associated to the body and another one representing the internal degrees of freedom. For the hydrogen atom $\Phi(x_r, x_{cm}) = \phi_r(x_r)\phi_{cm}(x_{cm})$ represents the *atomic limit.*

The election of the centre of mass, *cm*, to represent the position of the body will be shown later to be consistent with the form of the relational kinetic energy introduced. The variable x_r (*r* for *relative* or *relational*) stands for $x_e - x_p$. Our constructive postulate is that when proton and electron are conceived as (spatially) separated entities, the laws of motion correspond to electromagnetism and when conceived as a hydrogen atom, the usual QM is somehow recovered[64]. In fact, the present approach is broader than the usual QM in at least two aspects: (a) The decomposition of the wave function as a product is not imposed, but rather regarded as a limit and (b) the proton is regarded as something more than just the (point charge) source of electrostatic potential. Hence, starting from the laws of EM, the structure of a general quantum theory for $|\Psi\rangle$ can be postulated as surpassing EM. Specialising this theory to the atomic case, the QM of the hydrogen atom is obtained.

5.3.1.2. *Integrals on operators.* We will call the value of the integral

$$\int d^3x_a \; d^3x_b \; \Big(\Psi^*(x_a,x_b)(\hat{O}\Psi(x_a,x_b))\Big) \equiv \Big\langle \Psi^*(\hat{O}\Psi)\Big\rangle \equiv \Big\langle \Psi^*|\hat{O}|\Psi\Big\rangle \qquad (5.3.1)$$

the *integrated value* of the operator $\hat{O}$, where $\hat{O}$ is an operator that maps the wave function Ψ as $\hat{O}\Psi(x_a,x_b) = \Psi'(x_a,x_b)$[65].

5.3.1.3. *Physical Background.* In this section our goal is to develop an electromagnetic theory of microscopic systems encompassing both QM and Maxwell's electrodynamics, starting with the principles stated at the beginning of this Section and in Chapter 4. Hence, we will compute Maxwell's EM interaction energy, either between the atom and external EM-fields or between different fields arising within the atom. We display first some classical results that are relevant for the coming computations.

Maxwell's electromagnetic *interaction energy* between entities 1 and 2 reads $\mathcal{E} = \int d^3x \left[\epsilon_0 E_1 \cdot E_2 + \frac{1}{\mu_0} B_1 \cdot B_2\right]$ (cf. eq. (5.2.3)). As such, this expression is independent of the choice of gauge for the electromagnetic potentials A, V. In the sequel, we will adopt the Coulomb gauge whenever required.

The vector and scalar potentials associated to a charge-current density $(Cq\rho, j)$ in a (electro- and magneto-) static situation are

$$\left(\frac{V}{C}, A\right)(x,t) \quad = \quad \frac{\mu_0}{4\pi}\int d^3y \, \frac{(Cq\rho, j)(y,t)}{|x-y|}.$$

The vector potential associated to a magnetisation M at the coordinate y reads:

$$A_M(x,t) \quad = \quad \frac{\mu_0}{4\pi}\nabla_x \times \int d^3y \, \frac{M(y,t)}{|x-y|}.$$

For potentials as defined in the Lorenz gauge there is a gauge transformation that brings them to Coulomb form i.e., ρ electrostatic and $\nabla \cdot A = 0$. Finally, electric and magnetic fields are obtained through Maxwell's equations $E = -\nabla V - \frac{\partial A}{\partial t}$ and $B = \nabla \times A$.

64 Conventional QM also assumes whenever necessary that $\psi_{cm}(x_{cm})$ corresponds to a pointlike particle at rest.

65 We will drop the time argument of the wave function whenever possible to lighten the notation.

5.3.1.4. *Electromagnetic static potentials.* We define the density within the atom as $\rho(x_a,x_b) = |\Psi(x_a,x_b)|^2$. When appropriate, protonic and electronic densities may be defined as $\int d^3x_b\, |\Psi(x_a,x_b)|^2 \equiv \rho_a(x_a)$ for each component, $a = e,p$. In the electromagnetic limit of full separation between electron and proton ($|\Psi\rangle = |\psi_a\rangle|\psi_b\rangle$), with a wave function for particle b normalised to unity, we recover $\rho_a(x_a) = |\psi_a(x_a)|^2$. Protonic and electronic current-densities within the atom are defined similarly for $a = e,p$ by

$$j_a(x_p,x_e) = -\frac{i\hbar q_a}{2m_a}\left(\Psi^*(x_p,x_e)\left(\nabla_a\Psi(x_p,x_e)\right) - \Psi(x_p,x_e)\left(\nabla_a\Psi^*(x_p,x_e)\right)\right)$$

with $q_e = -q_p$.

The electrostatic potential reads

$$V_a(x,t) \quad = \quad \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, d^3x_b\, \frac{1}{|x-x_a|}\,|\Psi(x_a,x_b)|^2 = \frac{q_a}{4\pi\epsilon_0}\langle\Psi|\frac{1}{|x-x_a|}|\Psi\rangle.$$

Moreover,

$$V_a(x,t) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, \frac{1}{|x-x_a|}\left(\int d^3x_b\,|\Psi(x_a,x_b)|^2\right) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, \frac{\rho_a(x_a)}{|x-x_a|}$$

showing that the electromagnetic structure is preserved in general, not just in the limit of full separation.

Similarly, the vector potential (Coulomb gauge) associated to an intrinsic microscopic magnetisation $M_a(x_a,x_b) = M_a(x_a)\rho(x_a,x_b)$ reads:

$$\begin{aligned}
A_{M_a}(x,t) &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_b\; d^3x_a\, \frac{M_a(x_a,x_b)}{|x-x_a|} \\
&= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_a\, \frac{M_a(x_a)}{|x-x_a|}\rho(x_a) \\
&= \frac{\mu_0}{4\pi}\nabla_x \times \langle\Psi|\frac{M_a}{|x-x_a|}|\Psi\rangle
\end{aligned}$$

(the index in ∇ indicates the derivation variable) where Ψ is the wave function. In the electromagnetic limit, the vector potential reads:

$$\begin{aligned}
A_{M_a}(x,t) &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_a\, \frac{M_a}{|x-x_a|}\,|\psi_a(x_a)|^2 \\
&= \frac{\mu_0}{4\pi}\nabla_x \times \langle\psi_a|\frac{M_a}{|x-x_a|}|\psi_a\rangle,
\end{aligned}$$

which is the classical EM potential for a density $|\psi_a(x_a)|^2$.

For a static current we have

$$A_a(x,t) \quad = \quad \frac{1}{2}\frac{\mu_0 q_a}{4\pi m_a}\int d^3x_a\, d^3x_b\left(\Psi^*(x_a,x_b)\frac{1}{|x-x_a|}(-i\hbar\nabla_a)\Psi(x_a,x_b) + c.c.\right)$$

where *c.c.* is complex conjugate). Here we may also define

$$\widetilde{j}_a(x_a) = \frac{q_a}{2m_a}\int d^3x_b\;\left(\Psi^*(x_a,x_b)(-i\hbar\nabla_a)\Psi(x_a,x_b)\right) + c.c.$$

and set

$$A_a(x,t) \quad = \quad \frac{\mu_0}{4\pi}\int d^3x_a \, \left(\frac{\widetilde{j}_a(x_a)}{|x-x_a|}\right).$$

5.3.1.5. *On atomic magnetic moments.* We associate an intrinsic magnetisation to proton and electron, following Pauli's original work [Pauli Jr, 1927]. A general form for the intrinsic magnetisation for a microscopic constituent a reads

$$M_a = \frac{q_a}{m_a}S_a$$

where $S_a = \dfrac{\hbar}{2}\sigma^a$ and σ^a are the Pauli matrices for constituent a.

5.3.2. Variational formulation. The action (or action integral) is the integral of the Lagrangian over time, namely:

$$\mathcal{L} = \int d^3y \left[\frac{1}{2}\left(\Psi^*\left(i\hbar\dot{\Psi}\right) + (i\hbar\dot{\Psi})^*\Psi\right) + \frac{\hbar^2}{2m}\Psi^*T\Psi + \frac{1}{2}\sum_{i\neq j}\left(\frac{1}{\mu_0}B_i\cdot B_j - \epsilon_0 E_i\cdot E_j\right)\right] \tag{5.3.2}$$

with $i,j \in \{e,p,L\}$ which stand for electron, proton and Laboratory. The symbol T stands for kinetic energy and its form will be presented later. Actually "d^3y" is a symbol indicating that the integral is performed over all non-temporal arguments of the wave function, which in the case of the hydrogen atom will be at least six.

At this point we are interested in the internal energies associated to stationary states of the microscopic entity (the Hydrogen atom). Beginning this Chapter we assumed that the microscopic entity did not have an internal structure and therefore there was no "self-energy". In the present situation the atomic energy will have an internal part and an interaction part, as implied by eq. (5.3.2).

We propose therefore that the stationary states of the hydrogen atom have an associated wave function $\Psi_\omega(x_e,x_p)$, where the indices e and p correspond to the inferred electron and proton, whose unifying entity is the atom. x represents appropriate arguments of the wave function, satisfying

$$(\hbar\omega\Psi_\omega) - \frac{\hbar^2}{2m}T\Psi_\omega - \hat{H}\Psi_\omega = 0 \tag{5.3.3}$$

(we drop the index ω in the sequel) where

$$\left\langle\delta\Psi\hat{H}_{EM}\Psi\right\rangle = \delta_{\Psi^*}\left\langle\frac{1}{\mu_0}\left(B_L\cdot(B_e+B_p)+B_e\cdot B_p\right)+\epsilon_0\left(E_L\cdot(E_e+E_p)+E_e\cdot E_p\right)\right\rangle$$

the variation of the wave function is $\delta\Psi$ and $\langle\cdots\rangle$ indicates integration over all coordinates as in eq. (5.3.1).

5.4. Contributions to the integrated value of H

5.4.1. Relational kinetic energy. In terms of classical mechanics, the kinetic energy associated to the hydrogen atom can be organised as follows. To fix ideas let index b correspond to the proton and a to the electron. The quantum mechanical wave function is $\Psi(x_a,x_b)$. The center-of-mass coordinate (relative to some external reference such as "the laboratory") and the relative coordinate (which is an invariant of the description, independent of an external reference) can be defined as usual, satisfying

$$\begin{aligned} x_{cm} &= \frac{m_a x_a + m_b x_b}{m_a + m_b} \\ x_r &= x_a - x_b \\ x_a &= x_{cm} + x_r \frac{m_b}{m_a + m_b} \\ x_b &= x_{cm} - x_r \frac{m_a}{m_a + m_b}. \end{aligned}$$

The wave function and the gradient (we also make use of $p_x = -i\hbar\nabla_x$) read:

$$\begin{aligned} \Psi(x_a, x_b) &= \Psi(x_{cm} + x_r \frac{m_b}{m_a + m_b}, x_{cm} - x_r \frac{m_a}{m_a + m_b}) = \Phi(x_{cm}, x_r) \\ \nabla_{cm} &= \nabla_a + \nabla_b \\ \nabla_r &= \frac{m_b}{m_a + m_b}\nabla_a - \frac{m_a}{m_a + m_b}\nabla_b \\ \Phi(x_{cm}, x_r) &= \phi(\frac{m_a x_a + m_b x_b}{m_a + m_b}, x_a - x_b) = \Psi(x_a, x_b) \\ \nabla_a &= \nabla_{cm}\frac{m_a}{m_a + m_b} + \nabla_r \\ \nabla_b &= \nabla_{cm}\frac{m_b}{m_a + m_b} - \nabla_r \end{aligned}$$

Hence,

$$\begin{aligned} T_a + T_b = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} &= -\hbar^2 \left(\nabla_{cm}^2 \frac{1}{2(m_a + m_b)} + \nabla_r^2 \frac{(m_a + m_b)}{2m_a m_b} \right) \\ &= \frac{-\hbar^2}{2(m_a + m_b)}\nabla_{cm}^2 + \frac{-\hbar^2}{2m_r}\nabla_r^2 = T_{cm} + T_r. \end{aligned}$$

The associated integrated values read:

$$\mathcal{E}_k = \langle\Psi|T_a|\Psi\rangle + \langle\Psi|T_b|\Psi\rangle \quad = \quad \langle\Phi|T_{cm}|\Phi\rangle + \langle\Phi|T_r|\Phi\rangle$$

5.4.2. Relative electrostatic potential energy. The Coulomb electrostatic interaction energy for an atomic density $|\Psi(x_a, x_b)|^2$ reads:

$$\begin{aligned} \mathcal{E}_C^{x_a, x_b} &= \epsilon_0 \int d^3x\, E_1 \cdot E_2\, |\Psi(x_a, x_b)|^2 = -\epsilon_0 \int d^3x\, E_1 \cdot \nabla V_2\, |\Psi(x_a, x_b)|^2 \\ &= \int d^3x\, \epsilon_0 \left[-\nabla \cdot (E_1 V_2) + (\nabla \cdot E_1)\, V_2\right] |\Psi(x_a, x_b)|^2 . \end{aligned}$$

The full energy is given by averaging over the wave function Ψ. The first contribution vanishes when integrated over all space by Gauss theorem, and the second term can be recast as $-\epsilon_0(\nabla \cdot \nabla V_1)V_2$, namely

$$\begin{aligned} \mathcal{E}_C &= \int d^3x_a\, d^3x_b\, |\Psi(x_a, x_b)|^2 \int d^3x\, \epsilon_0 \left(\frac{q_a q_b}{(4\pi\epsilon_0)^2}\right) \left(-\frac{1}{|x - x_b|}\Delta\frac{1}{|x - x_a|}\right) \\ &= -\frac{e^2}{4\pi\epsilon_0} \int d^3x_e\, d^3x_p\, |\Psi(x_e, x_p)|^2 \frac{1}{|x_e - x_p|} = -\langle\Psi|\frac{e^2}{4\pi\epsilon_0}\frac{1}{|x_e - x_p|}|\Psi\rangle \end{aligned}$$

where $q_a = e = -q_b$ and e is the electron charge (negative), i.e., a is the electron, b the proton. In the electromagnetic limit it corresponds to the potential energy between two separate charge densities $\rho_{k=}|\psi_k(x_k)|^2$ of opposite sign, corresponding to Maxwell's equation $q_k\rho_k = \epsilon_0 \nabla \cdot E_k$.

5.4.3. Interaction of relative current with external magnetic field. For this interaction *relative current* means the currents (j_a, j_b) of both electron and proton regarding their motion relative to the (source of) external field. Recall that we associate $j = qv = \frac{q}{m}p = \frac{q}{m}(-i\hbar\nabla)$, i.e., current as a property of charge in (perceived) motion. The electromagnetic energy in the external field $B(x)$ is

$$\begin{aligned}
\mathcal{E}_{jB} &= \frac{1}{\mu_0}\int d^3x_a\ d^3x_b\ \Psi(x_a,x_b)^*\int d^3x\, B(x)\cdot\nabla\times(A_a + A_b)\,\Psi(x_a,x_b) \\
&= \frac{1}{2}\frac{1}{\mu_0}\frac{\mu_0}{4\pi}\int d^3x_a\ d^3x_b\ \Psi(x_a,x_b)^*\int d^3x\, B(x)\cdot\nabla\times\left(\frac{1}{|x-x_a|}\frac{q_a}{m_a}p_a\right)\Psi(x_a,x_b) + c.c. \\
&+ \frac{1}{2}\frac{1}{\mu_0}\frac{\mu_0}{4\pi}\int d^3x_a\ d^3x_b\ \Psi(x_a,x_b)^*\int d^3x\, B(x)\cdot\nabla\times\left(\frac{1}{|x-x_b|}\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c.
\end{aligned}$$

Having in mind the (normal) Zeeman effect, we perform an explicit calculation using that the vector potential for the approximately constant external field B reads $B \equiv \nabla\times A_L = \nabla\times\left(\chi(x)\frac{1}{2}x\times B\right)$, with χ a smoothed version of the characteristic function for the experiment's region. We place the origin of coordinates inside the apparatus and assume the characteristic function to be 1 inside a macroscopically large ball K around the origin, so that

$$\int_{K\times K} dx_a\ dx_b\ |\Psi(x_a,x_b)|^2 \simeq 1$$

(i.e., the atom is inside the measurement device). In such a case,

$$\begin{aligned}
\mathcal{E}_{jB} &\sim \frac{1}{2}\int d^3x_a\ d^3x_b\ \Psi(x_a,x_b)^*\left(\frac{1}{2}x_a\times B\cdot\frac{q_a}{m_a}p_a + \frac{1}{2}x_b\times B\cdot\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c. \\
&= -\frac{1}{2}\int d^3x_a\ d^3x_b\ \Psi(x_a,x_b)^*\ \frac{1}{2}B\cdot\left(x_a\times\frac{q_a}{m_a}p_a + x_b\times\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c. \\
&= -\frac{e}{2m_r}\langle B\cdot\left(\frac{m_r}{m_e+m_p}x_r\times p_{cm} + x_{cm}\times p_r + \frac{m_p-m_e}{m_p+m_e}L_r\right)\rangle \\
&\sim -\left(\frac{e}{2m_e} + O(\frac{m_e}{m_p})\right)\langle\Psi|B\cdot L_e|\Psi\rangle
\end{aligned}$$

where in the last line we have switched to cm, r coordinates and used that $\langle p_r\rangle = \langle p_{cm}\rangle = 0$ and $[x_r, \nabla_{cm}] = [x_{cm}, \nabla_r] = 0$.

Further, the argument of the above integral reads,

$$\nabla\times A_L\cdot\nabla\times A_a = \nabla\cdot(A_L\times(\nabla\times A_a)) + A_L\cdot\nabla\times(\nabla\times A_a)\,.$$

The first term vanishes by Gauss theorem. The experimental conditions assume that the atom lies well inside the region of magnetic field and that the border effects $\nabla\chi(x-x_{CM})\times\left(\frac{1}{2}x\times B\right)$ can be disregarded. Moreover, $\nabla\times(\nabla\times A_a) = \nabla(\nabla\cdot A_a) - \Delta A_a$. For the first term, we integrate over x_a, x_b, using that $\nabla_x\frac{1}{|x-x_a|} = -\nabla_{x_a}\frac{1}{|x-x_a|}$, arriving to the integral of

$\frac{1}{|x-x_a|}\nabla_{x_a}\cdot\left(\Psi(x_a,x_b)^*\frac{q_a}{m_a}p_a\Psi(x_a,x_b)\right)+c.c.$, which is zero for an atom in a stationary state, since by the continuity equation it corresponds to the time-derivative of the atomic charge density. For the other term, recall that $-\Delta\frac{1}{|x-x_a|}=4\pi\delta(x-x_a)$. Hence,

$$\begin{aligned}
\mathcal{E}_{jB} &= \frac{1}{2}\int d^3x_a\,d^3x_b\,\Psi(x_a,x_b)^*\int d^3x\,A_L\cdot\left(\delta(x-x_a)\frac{q_a}{m_a}p_a+\delta(x-x_b)\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)\\
&\quad +c.c.\\
&= \frac{1}{2}\int d^3x_a\,d^3x_b\,\Psi(x_a,x_b)^*\left(A_L(x_a)\cdot\frac{q_a}{m_a}p_a+A_L(x_b)\cdot\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)+c.c.\\
&\sim \frac{1}{2}\int d^3x_a\,d^3x_b\,\Psi(x_a,x_b)^*\left(\frac{1}{2}x_a\times B\cdot\frac{q_a}{m_a}p_a+\frac{1}{2}x_b\times B\cdot\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)+c.c.
\end{aligned}$$

assuming that the atomic wave function is negligibly small outside the the region hosting the measuring device. We now move to center-of-mass and relative coordinates, assuming further that the atom as a whole is at rest (at the centre of mass location) during the experiment (i.e., $\langle\Psi|p_{CM}|\Psi\rangle=0$, $p_r\sim p_e$, $x_r\sim x_e$ and $\frac{m_p-m_e}{m_p+m_e}\sim 1$). Hence,

$$\begin{aligned}
\mathcal{E}_{jB} &\sim \frac{1}{2}\left\{\left\langle\Psi(x_a,x_b)^*\left(\frac{1}{2}(x_a-x_{CM})\times B\cdot\frac{q_a}{m_a}\right)\Psi(x_a,x_b)\right\rangle\right.\\
&\quad\left.+\left\langle\Psi(x_a,x_b)^*\left(\frac{1}{2}(x_b-x_{CM})\times B\cdot\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)\right\rangle\right\}+c.c.\\
&= \frac{1}{2}\left(-\frac{e}{2(m_p+m_e)}\right)\left\langle\Psi(x_{CM},x_r)^*\,B\cdot\left(\frac{m_p}{m_e}-\frac{m_e}{m_p}\right)L_r\Psi(x_{CM},x_r)\right\rangle+c.c.\\
&= -\frac{e}{2m_r}\frac{m_p-m_e}{m_p+m_e}\langle\Psi|B\cdot L_r|\Psi\rangle\sim-\left(\frac{e}{2m_e}+O(\frac{m_e}{m_p})\right)\langle\Psi|B\cdot L_e|\Psi\rangle
\end{aligned}$$

5.4.4. Interaction of spin(s) with external magnetic field. We regard spin as an intrinsic magnetisation $\frac{q_a}{m_a}S_a$, leading to a vector potential

$$A_s(x,t) = \frac{\mu_0}{4\pi}\nabla\times\langle\Psi|\left(\frac{q_a}{m_a}\frac{S_a}{|x-x_a|}-\frac{q_a}{m_b}\frac{S_b}{|x-x_b|}\right)|\Psi\rangle \qquad (5.4.1)$$

whose interaction energy with an external magnetic field $B(x,t)$ reads

$$\begin{aligned}
\mathcal{E}_{SB} &= \frac{1}{\mu_0}\int d^3x\;B(x,t)\cdot\nabla\times A_s(x,t)\\
&= \frac{e}{4\pi}\langle\Psi|\int d^3x\;B(x,t)\cdot\left(\nabla\times\nabla\times\left(\frac{1}{m_e}\frac{S_e}{|x-x_e|}-\frac{1}{m_p}\frac{S_p}{|x-x_p|}\right)\right)|\Psi\rangle\\
&= \frac{e}{4\pi}\langle\Psi|\int d^3x\;B(x,t)\cdot\left[\nabla\left(\nabla\cdot\left(\frac{1}{m_e}\frac{S_e}{|x-x_e|}-\frac{1}{m_p}\frac{S_p}{|x-x_p|}\right)\right)\right]|\Psi\rangle\\
&- \frac{e}{4\pi}\langle\Psi|\int d^3x\;B(x,t)\cdot\left(\Delta\left(\frac{1}{m_e}\frac{S_e}{|x-x_e|}-\frac{1}{m_p}\frac{S_p}{|x-x_p|}\right)\right)|\Psi\rangle
\end{aligned}$$

Since $\nabla \cdot B = 0$, the first contribution integrates to zero by Gauss theorem since $B \cdot \nabla\phi = \nabla \cdot (B\phi) - \phi\nabla \cdot B$. Hence,

$$\mathcal{E}_{SB} = e\langle\Psi| \left(\frac{1}{m_e} B(x_e,t) \cdot S_e - \frac{1}{m_p} B(x_p,t) \cdot S_p \right) |\Psi\rangle$$

thus completing the description of Zeeman effect. In the atomic limit the usual expression is recovered.

Stern-Gerlach effect. All forces, including the Lorentz force [Solari and Natiello, 2022b], are obtained from actions as the response of the action integral to a variation of the relative position of the interacting bodies. Displacing by δx the position of the field B one gets

$$\delta\mathcal{E}_{SB} = e\langle\Psi| \left(\frac{1}{m_e} B(x_e+\delta x,t) \cdot S_e - \frac{1}{m_p} B(x_p+\delta x,t) \cdot S_p \right) |\Psi\rangle = \delta x \cdot F$$

and then $F \sim \langle\Psi|\nabla \left(\frac{e}{m_e} B(x,t) \cdot S \right) |\Psi\rangle$. The force is nonzero only for spatially varying magnetic fields as observed in the experiment [Bauer, 2023].

5.4.5. Spin-orbit interaction. If we follow the standard discourse of quantum physics, spin-orbit represents mainly the interaction between the electron's orbit and spin. Such heuristic approach would break one of our fundamental propositions: all the energies in the hydrogen atom are interaction energies between proton and electron, no self energy is involved. A relational view is actually forced to recognise that the orbit of the electron is a motion relative to the proton. Hence, only their relative velocity can matter. Such observation does not solve our problem, ... but let us follow its lead.

The coupling of spin and relative current has two parts, namely the interaction of proton spin $\frac{-e}{m_p} S_p$ with the relative current $\frac{e p_r}{m_r}$ as perceived by the proton and the corresponding interaction of electron spin $\frac{e}{m_e} S_e$ with relative current $\frac{-e(-p_r)}{m_r}$ as perceived by the electron. $m_r = \frac{m_e m_p}{m_e + m_p}$ stands for the reduced mass and $e\left(\frac{p_e}{m_e} - \frac{p_p}{m_p}\right) = e\frac{p_r}{m_r}$. The magnetic field operator associated to the relative current as seen by the proton is hence the curl of the vector potential operator associated to that current (recall that the energy contribution is the integrated value of the operators over the wave function), and correspondingly for the current perceived by the electron:

$$\begin{aligned}
\hat{B}_{jp}(x)|\Psi\rangle &= \frac{\mu_0 e}{4\pi} \nabla_x \times \left(\frac{1}{|x-x_p|} \left(\frac{p_e}{m_e} - \frac{p_p}{m_p} \right) \right) |\Psi\rangle \\
&= \frac{\mu_0 e}{4\pi} \nabla_x \frac{1}{|x-x_p|} \times \left(\frac{p_e}{m_e} - \frac{p_p}{m_p} \right) |\Psi\rangle \qquad (5.4.2) \\
\hat{B}_{je}(x)|\Psi\rangle &= \frac{\mu_0 e}{4\pi} \nabla_x \frac{1}{|x-x_e|} \times \left(\frac{p_e}{m_e} - \frac{p_p}{m_p} \right) |\Psi\rangle .
\end{aligned}$$

The magnetic field operators associated to the spin are:

$$\begin{aligned}\hat{B}_{se}(x) &= \frac{\mu_0 e}{4\pi}\nabla_x \times \left(\nabla_x \times \left(\frac{1}{|x-x_e|}\frac{S_e}{m_e}\right)\right)\\ \hat{B}_{sp}(x) &= -\frac{\mu_0 e}{4\pi}\nabla_x \times \left(\nabla_x \times \left(\frac{1}{|x-x_p|}\frac{S_p}{m_p}\right)\right)\end{aligned}$$

and the energy contribution is

$$\mathcal{E}_{SO} = \kappa\frac{1}{\mu_0}\langle\Psi|\int d^3x\ \hat{B}_{se}(x)\cdot\hat{B}_{jp}(x) + \hat{B}_{sp}(x)\cdot\hat{B}_{je}(x)|\Psi\rangle$$

where κ is a numerical constant that needs to be determined. We transform the spin field operators as

$$\begin{aligned}\hat{B}_{se}(x)\cdot\hat{B}_{jp}(x) &= \frac{\mu_0 e}{4\pi m_e}\hat{B}_{jp}(x)\cdot\left(\nabla\left(\nabla\cdot\frac{S_e}{|x-x_e|}\right) - \Delta\left(\frac{S_e}{|x-x_e|}\right)\right)\\ &= \frac{\mu_0 e}{4\pi m_e}\left(\nabla\cdot\left(\hat{B}_{jp}(x)\left(\nabla\cdot\frac{S_e}{|x-x_e|}\right)\right) - \hat{B}_{jp}(x)\cdot\Delta\left(\frac{S_e}{|x-x_e|}\right)\right)\\ &= \frac{\mu_0 e}{4\pi m_e}\hat{B}_{jp}(x)\cdot(4\pi S_e\delta(x-x_e))\\ \hat{B}_{sp}(x)\cdot\hat{B}_{je}(x) &= -\frac{\mu_0 e}{4\pi m_p}\hat{B}_{je}(x)\cdot(4\pi S_p\delta(x-x_p))\end{aligned} \tag{5.4.3}$$

using Gauss theorem and that $\nabla\cdot B = 0$. Performing the $x-$integral first, we obtain

$$\begin{aligned}\mathcal{E}_{SO} &= -\kappa\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{-(x_p-x_e)}{|x_e-x_p|^3}\times\frac{p_r}{m_r}\right)|\Psi\rangle\\ &= -\kappa\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{L_r}{m_r|x_e-x_p|^3}\right)|\Psi\rangle,\end{aligned} \tag{5.4.4}$$

where $L_r = x_r \times p_r$ is the relative angular momentum operator.

Let us complete the expression determining the value of κ before we turn back to the question that opens this subsection. The expression for the electromagnetic energy (eq. (5.2.3)) was introduced by Maxwell considering the kind of interactions known at his time. It is symmetric in the indexes of the two interacting systems, hence, if $\mathcal{P}(12)$ is a permutation of indexes, the energy can be written as:

$$\mathcal{E}_{SO} = \frac{1}{2}\left\{\int d^3x\left[\epsilon_0 E_1\cdot E_2 + \frac{1}{\mu_0}B_1\cdot B_2\right] + \mathcal{P}(12)\int d^3x\left[\epsilon_0 E_1\cdot E_2 + \frac{1}{\mu_0}B_1\cdot B_2\right]\right\}.$$

We may call the first integral the way in which system one acts upon system two and the second integral is the reciprocal action. The energy can then be obtained by first establishing one action and next "symmetrising". The action of $\mathcal{P}(12)$ is merely changing the point of view in a somewhat arbitrary form, while the imposition of symmetry removes the arbitrariness since the acting group is a group of arbitrariness (see [Solari and Natiello, 2018]). The operation of taking two different view points corresponds to operating with $\mathcal{P}(ep)$ as it can be easily verified in all the previous expressions. Hence, unless $\kappa = \frac{1}{2}$ we would not be counting the interaction properly.

It remains to show that we are dealing with an interaction between two different entities and not an internal interaction. Consider our final expression, eq. (5.4.4) and write it as:

$$\begin{aligned}
\mathcal{E}_{SO} &= -\frac{1}{2}\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{(x_e-x_p)\times(v_e-v_p)}{|x_e-x_p|^3}\right)|\Psi\rangle \\
&= \frac{1}{2}\int d^3x_e d^3x_p\left\{\Psi^\dagger\left(\frac{(x_e-x_p)}{|x_e-x_p|^3}\right)\cdot\frac{e^2}{4\pi\epsilon_0 C^2}\left(\frac{S_e\times v_e}{m_e}-\frac{S_p\times v_p}{m_p}\right)\Psi\right\} \\
&\quad -\frac{1}{2}\int d^3x_e d^3x_p\left\{\Psi^\dagger\left(\frac{(x_e-x_p)}{|x_e-x_p|^3}\right)\cdot\frac{\mu_0 e^2}{4\pi}\left(\frac{S_e\times v_p}{m_e}-\frac{S_p\times v_e}{m_p}\right)\Psi\right\}
\end{aligned}$$

($\Psi^\dagger$ stands for the row matrix that is the transpose and complex conjugate of Ψ). Consider further the case $\Psi=\psi_e\psi_p$ and in addition the distance $|x_e-x_p|$ is macroscopic. In such a case we can consider that the variation of the relative position with respect to $\langle x_e - x_p\rangle$ is negligible in front of the macroscopic distance. The main contribution of the first term reads

$$\sim \frac{1}{2}\frac{1}{4\pi\epsilon_0 C^2}\frac{(x_p-x_e)}{|x_e-x_p|^3}\cdot\int d^3x_e\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)\int d^3x_p\left(-e|\psi_p|^2\right),$$

which represents the interaction between an electric dipole $\int d^3x_e\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)$ located in the electron (now approximated as a point) and the proton (approximated as a point in front of the macroscopic distance). This view is compatible with the idea that magnetic dipoles in motion produce electric dipole fields. Up to a certain point, $\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)$ represents a density of electric dipoles and $-e|\psi_p|^2$ a density of charge. The difference with such densities is that it is not possible to limit the interaction to "part of the electron" or "part of a proton", hence a integration over full space is always mandatory., thus the view of $\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)$ as density of classical dipoles is only an analogy, it leaves aside the unity of the electron.

Actually, the dominant term in spin-orbit interactions corresponds to the electric dipole of the electron acting upon the proton. Thus, what was described as an interaction between the electron and its own orbit is now identified as the action of the electric dipole associated with the moving electron with the proton charge. An equivalent contribution appears with the action of $\mathcal{P}(ep)$. The last two terms are magnetic field interactions between proton and electron.

The expression 5.4.4 corresponds well with the final expression in textbook derivations. However, we have not resorted to analogy, nor have we patched this or any other expression with gyromagnetic numbers and have not further patched the expression with "relativistic corrections" (such as Thomas' correction) in the need to agree with experiments. The derivation of the spin-orbit contribution highlights the differences between the utilitarian/instrumentalist and the old style approach.

Related experiments. One of the best known results of the atomic limit was the prediction of the hydrogen spectroscopic lines with Schrödinger's wave equation, providing a full theoretical expression of the Rydberg constant. The transition $n=2$ to $n=1$ [Kramida et al., 2023] is reported as the spectroscopic line at

$\lambda = 1215.6699\mathring{A}$. The line-width allows for a calculated resolution into $\lambda_1 = 1215.668237310\mathring{A}$ and $\lambda_2 = 1215.673644608\mathring{A}$ The state $n = 2$ is an octuplet, partially resolved by the spin-orbit Hamiltonian of Section 5.4.5 into a quadruplet and a doublet for $l = 1$ and two unresolved states with $l = 0$. The difference between the two energy levels is calculated as

$$\begin{aligned}
\Delta E_{SO} &= \frac{3}{2}\hbar^2 \frac{1}{24a_0^3}\left(\frac{1}{2}\frac{\mu_0 e^2}{4\pi m_e^2}\right)\frac{m_e}{m_r} \\
&= \frac{3}{96}\frac{m_e}{m_r}\alpha^4 m_e C^2 \\
&= 7.259023470408092 \cdot 10^{-24} J
\end{aligned}$$

while the calculated experimental energy difference amounts to $\Delta E_{exp} = hC(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) = 7.26816814178113 \cdot 10^{-24} J$. The contribution consists of: $\frac{3}{2}\hbar^2$ coming from the level difference in $\langle L \cdot S\rangle$ between $j = 3/2$ and $j = 1/2$ (for $l = 1$ and $s = 1/2$), $\frac{1}{24a_0^3} = \langle R_{21}(r)|\frac{1}{r^3}|R_{21}(r)\rangle$ for the associated Hydrogen (n, l)-radial wave function, $a_0 = \frac{\hbar}{\alpha m_e C}$ is the Bohr radius, m_e is the electron mass and $\alpha = \frac{\mu_0 e^2 C}{4\pi\hbar}$ is the fine-structure constant. We have disregarded the nuclear spin contribution.

5.4.6. Spin-spin interaction (part of hyperfine structure). The interaction between spins within the atom uses an expression for the vector potential corresponding to that in Section 5.4.4 and reads

$$\begin{aligned}
\mathcal{E}_{SS} &= \frac{1}{\mu_0}\langle\Psi|\int d^3x\ B_e(x,t)\cdot B_p(x,t)|\Psi\rangle \\
B_e &= \nabla\times\left(\nabla\times\left(\frac{e}{m_e}\frac{\mu_0}{4\pi}\frac{S_e(x_e)}{|x-x_e|}\right)\right) \\
B_p &= \nabla\times\left(\nabla\times\left(\frac{-e}{m_p}\frac{\mu_0}{4\pi}\frac{S_p(x_p)}{|x-x_p|}\right)\right)
\end{aligned}$$

To proceed with the calculation we need to break the $e-p$ symmetry, namely $\mathcal{E}_{SS} = \frac{1}{2}\left(\mathcal{E}_{SS}^{e\to p} + \mathcal{E}_{SS}^{p\to e}\right)$. We pick the electron to proceed with the calculation, applying Gauss theorem again as in Section 5.4.4:

$$\begin{aligned}
\mathcal{E}_{SS}^{e\to p} &= \frac{1}{\mu_0}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\int d^3x\ B_p(x)\cdot\left(-\Delta\frac{S_e(x_e)}{|x-x_e|}\right)|\Psi\rangle \\
&= -\frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\int d^3x\ \left(\nabla_x\times\left(\nabla_x\times\frac{S_p}{|x-x_p|}\right)\right)\cdot S_e\delta(x-x_e)|\Psi\rangle \\
&= -\frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left(\nabla_x\times\left(\nabla_x\times\frac{S_p}{|x-x_p|}\right)\right)\cdot S_e\right]_{x=x_e}|\Psi\rangle
\end{aligned}$$

Through the vector calculus identities

$$\begin{aligned} -\left(\nabla_x \times \left(\nabla_x \times \varphi(x) S_p\right)\right) \cdot S_e &= \left(S_e \times \nabla_x\right) \cdot \left(S_p \times \nabla_x\right) \varphi(x) \\ \left(S_e \times \nabla_x\right) \cdot \left(S_p \times \nabla_x\right) \frac{1}{|x - x_p|} &= \left[\left(S_e \cdot S_p\right) \Delta_x - \left(S_e \cdot \nabla_x\right) \left(S_p \cdot \nabla_x\right)\right] \frac{1}{|x - x_p|} \end{aligned}$$

we obtain

$$\begin{aligned} \mathcal{E}_{SS}^{e \to p} &= \frac{e}{m_p} \frac{e}{m_e} \frac{\mu_0}{4\pi} \langle \Psi | \left[\left[\left(S_e \cdot S_p\right) \Delta_x - \left(S_e \cdot \nabla_x\right) \left(S_p \cdot \nabla_x\right) \right] \frac{1}{|x - x_p|} \right]_{x = x_e} |\Psi\rangle \\ \mathcal{E}_{SS} &= \frac{e}{m_p} \frac{e}{m_e} \frac{\mu_0}{4\pi} \langle \Psi | \left[\left[\left(S_e \cdot S_p\right) \Delta_x - \left(S_e \cdot \nabla_x\right) \left(S_p \cdot \nabla_x\right) \right] \frac{1}{|x - x_e|} \right]_{x = x_p} |\Psi\rangle \end{aligned}$$

since both contributions are identical. The spin-spin interaction is symmetric in front of the interchange of electron and proton and then carries a factor one in front.

5.4.7. Interaction with an external electric field. Suppose the atom is in interaction with an external electric field, E_Z. The electromagnetic interaction is then:

$$\mathcal{E}_{AZ} = \epsilon_0 \int dx \ \left(\left(E_A + E_A^{am}\right) \cdot E_Z\right)$$

where $E_A = -\nabla V$ and $E_A^{am} = -\frac{\partial A}{\partial t}$ being (V, A) the electromagnetic potentials associated to the atom:

$$\begin{aligned} A &= \frac{\mu_0}{4\pi} \nabla \times \langle \Psi | \left(\frac{e}{m_e} \frac{M_e}{|x - x_e|} - \frac{e}{m_p} \frac{M_p}{|x - x_p|} \right) |\Psi\rangle \\ V &= \frac{e}{4\pi\epsilon_0} \langle \Psi | \left(\frac{1}{|x - x_e|} - \frac{1}{|x - x_p|} \right) |\Psi\rangle. \end{aligned}$$

Quantum Mechanics and Spectroscopy

The present Section has dealt with deriving the contributions to the quantum entity's energy (the hydrogen atom), which in the atomic limit correspond to the discrete states. As mentioned in Subsection 5.2.3, while we have not worked out the details of the interaction between the quantum entity and the EM field, it is clear that so far there has been no need to request the quantification of the EM field. Experimental observations do indeed show that the exchange of energy in absorption and emission processes is quantified. Yet, this fact does not imply that the field itself needs to be quantified. This observation extends also to the photoelectric effect.

5.5. Quantum mechanics and instrumentalism

The production of QM (a name attributed to Max Born [Born, 1955]) under the leadership of the Copenhagen school was embraced in an instrumentalist philosophical approach. Historically, the development was guided by the correspondence with classical mechanics (although it deals almost completely with electromagnetic interactions). Two of the three central productive tools in the creation of QM are: imagination (as it is illustrated by Bohr's well known planetary atom) and analogy of forms, i.e., the proposal of mathematical relations produced in a different context and imported into QM without satisfying the conditions necessary for their meaningful production[66].

The third element is the adoption of instrumentalism. The Copenhagen school was able to make important advances in QM introducing a matrix formalism, but as Born confessed in his Nobel lecture: "What the real significance of this formalism might be was, however, by no means clear" [Born, 1955]. The situation is not a surprise since instrumentalism is focused on (predictive) success rather than on significance. Its approach is more akin to technology than to science. It relies on trial and error, success or failure in reproducing and collating experimental results. The process of patching contradictions or inconsistencies with ad-hoc hypotheses makes refutation almost impossible. In so doing, it puts an undue pressure over the experimental side, since it requires experiments each day more complex, scarce and in need of interpretation. Compensating refutations by introducing ad-hoc patches leads more often than not to specialism, since different patches need not be mutually consistent. Seeking for the unity of knowledge goes precisely in the opposite direction. The explosion of Science into multiple sciences, each one constantly fragmenting in a fractal design, is not a consequence of the complexity of nature but rather the consequence of sacrificing reason in the altar of productivity.

The "correspondence principle", in the context of the utilitarian development, is what remains of the idea of cognitive surpass (see Footnote 13 in Foreword) where the old becomes a particular instance of the new generalisation.

In terms of the correspondence with classical mechanics, the idea of point particles is central to the statistical QM description. The concept of particle is so strongly associated to the extreme idealisation as points that, quite often, their point character is not mentioned until late in the works. If particles in classical mechanics, when possible, are idealised as mathematical points, particles in QM *are* mathematical points, the localisation in space becomes an essential property of what in conventional (textbook) QM is considered a particle.

Point particles are not an invention of the Copenhagen school, they are present as idealisations in almost every textbook of physics. In his theory of the electron, Lorentz [Lorentz, 1904] considered the electron as a point surrounded by a cloud of electricity located in the nearby ether. Further back in time, Faraday had opposed these views in his theory of vibrating rays [Faraday, 1855, p. 447] indicating that neither the material point nor the ether were needed if we considered matter as an inference arising as a consequence of the observation of EM action. Faraday recommended extreme prudence in introducing such inventions since in the past these metaphysical elements had proven to be difficult to remove.

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[66] For example: the Hamiltonian for a charged (point) particle used in classical mechanics, as can be seen in [Pauli Jr, 1927, Dirac, 1928], or Thomas' relativistic correction to the spin-orbit energy [Thomas, 1926] where, in order to patch Zeeman's formula obtained by analogy and refuted by experiments, Thomas applies Lorentz transformations relating two systems which are *not* in uniform motion with constant velocity with respect to each other (hence, the electromagnetic fields described in each system are *not* related by Lorentz transformations).

The scene changed with the irruption of the works by de Broglie [de Broglie, 1923, 1924] and Schrödinger [Schrödinger, 1926] who introduced the wave formulation of QM, although both of them relied in part on point particles. Later, Schrödinger strongly opposed this view.

As mentioned in the Introduction, the EM interactions in quantum mechanics were at variance with electrodynamics. The expected radiation of an electron orbiting the proton in an hydrogen-like atom was suppressed by ruling that no such radiation occurs for some special orbits. In short, conventional EM did not apply to quantum phenomena. The unity of physics was broken as it was broken a few years earlier when special relativity and its immaterial ether [Einstein, 1924] was accepted by the guild of physics as the cornerstone of electromagnetism, thus breaking with Newtonian mechanics.

The self-appraisal of instrumentalism perceives these cognitive decisions as realism or pragmatism and not as deficits. Instrumentalism advances without retracing its steps, without reconsidering the possibility of being in error, patching "theories" and narrowing their scope as needed to avoid refutation. Actually, when the word "physicist" was coined [Yeo, 1993] the increasing specialisation and lack of unity in the advances of physics was the main concern for philosophers of Nature such as Faraday and Whewell.

The uncomfortable situation of the wave-particle duality forced by Schrödinger and de Broglie is left behind by Max Born who introduces the (now) standard interpretation of wave functions as probability waves. As Schrödinger points out, this appears to be the only form of reconciliation with point particles that is at hand, and it is the form in which QM has been socially transmitted to several generations of physicists. Adopting this interpretation forces us to write the wave function in a basis of functions, $|o\rangle$, where the operator $\hat{O}$ representing the measurable property is diagonal, $\hat{O}|o\rangle = o|o\rangle$, with $o \in \mathbb{R}$, so the "matrix" representing the density of states, $\langle o|\Psi\rangle\langle\Psi|o'\rangle$, has only positive values in the diagonal and zero otherwise. Since two non commuting matrices (operators) cannot be simultaneously diagonalised, it is not possible to obtain probabilities for their joint measurement. The theory then ruled that this impossibility is an imposition of Nature, although in deriving it we have only used our interpretation. This difficulty in separating what comes from the subject (the observer) and what comes from the object (the observed) has been central to the criticism of natural science by Goethe, Faraday, Whewell and Peirce. It is the attitude in front of the (possible) error what makes the most striking difference between the instrumentalist approach and the philosophical approach. The instrumentalist scientist resorts to patches, restrictions and limitations. The final outcome is posed by S. Carroll as "Physicists do not understand quantum mechanics. Worse, they don't seem to want to understand it" [Carroll, 2019].

In a constructivistic perspective, theories are constructs and their interpretations must be totally in accordance with the abduction process employed in the construction stage, since abduction and interpretation form the two components of the phenomenological map (see Chapter 2). In the constructivistic sense, to "understand a theory" implies to be able to construct it rationally starting from observations and experimental information through a process rooted in previous knowledge (which might require adjustments under the light of the new information) and using synthetic judgments which are put to test by the consequences of the theory. QM has not been reconstructed under the idea of being a statistical theory. We have then the right to ask: where was the statistics introduced when all our considerations have been about the interactions of point particles? The resolution of this conflict is simple for the instrumentalist, they tell us: "the construction is not part of science" or even more strongly: science is not about the reality but just a matter of economy of thought.

The view of inferred matter as having identity by itself with forces that abut around was the standard view used in the construction of physics in the late XIXth Century [[529], Maxwell, 1873] and propagates there on, without being given reconsideration. For Einstein [Einstein, 1936] the concept of material point is fundamental for Newton's mechanics (not just an idealisation). In his work Einstein criticises Lorentz' proposal of electrons extending in the space (with soft borders in [Lorentz, 1892, §33] and spherical in [Lorentz, 1904, §8]) in the form:

> The weakness of this theory lies in the fact that it tried to determine the phenomena by a combination of partial differential equations (Maxwell's field equations for empty space) and total differential equations (equations of motion of points), which procedure was obviously unnatural. The unsatisfactory part of the theory showed up externally by the necessity of assuming finite dimensions for the particles in order to prevent the electromagnetic field existing at their surfaces from becoming infinitely great. The theory failed moreover to give any explanation concerning the tremendous forces which hold the electric charges on the individual particles. [Einstein, 1936]

The criticism is faulty in several accounts. As we have shown, a single mathematical entity, Hamilton's principle, is able to produce the equations of electromagnetism and mechanics. Einstein qualifies Lorentz' electron theory as "unsatisfactory" for using finite-size particles, since for Einstein forces (electromagnetic fields) exist in the surface of particles. Simply, he ignores Faraday, and furthermore, to be mathematical points seems to be a requirement for particles. Einstein gives no argument to justify the character of "unnatural" for Lorentz' procedure. Actually, "unnatural" appears to stand for "alien to my scientific habit". In contrast, mathematical statements require proofs, which we offered. The argument concerning a need for a force to held together the electron is equally faulty, it assumes that there are parts that need to be held together, that the quantum entity is a composite in which one part can act on another part. Acquaintance with Maxwell's arguments, make us to realise that the electrostatic energy computed in electromagnetism requires the possibility of dividing the charge in infinitesimal amounts. Something that can be considered a reasonable limit in macroscopic physics (Maxwell's situation) but is ridiculous when it comes to the electron.

Maxwell's concept of science and our own concept differ radically from Einstein's. For Maxwell and the present work, the construction is a relevant part of the theory, for Einstein:

> There is no inductive method which could lead to the fundamental concepts of physics. Failure to understand this fact constituted the basic philosophical error of so many investigators of the nineteenth century. It was probably the reason why the molecular theory, and Maxwell's theory were able to establish themselves only at a relatively late date. Logical thinking is necessarily deductive; it is based upon hypothetical concepts and axioms. How can we hope to choose the latter in such a manner as to justify us in expecting success as a consequence? [Einstein, 1936].

Einstein's utterly instrumentalist statements contrast with the legend of his early reading of Kant's "Critique of pure reason" which elaborates in the opposite direction through the concept of *synthetic judgment*. The final question posed was answered by Peirce's abduction (retroduction) [Peirce, 1994] and the whole issue was considered in Whewell's philosophy of science [Whewell, 2016], also here in contrasting terms.

In contrast, Einstein accepted without criticism that "Everywhere (including the interior of ponderable bodies) the seat of the field is the empty space. [Einstein, 1936]" (referring to the EM fields) despite being conscious that the idea perpetuated the essence of the ether [Einstein, 1924].

The reason we emphasise Einstein's positions is that he has been socially instituted as the "most perfect scientist", something "known" even by the illiterate. Actually, acceptance of his statement requires the previous acceptance of the utilitarian view of science.

Concluding the historical revision, the adoption of an instrumentalist approach weakens the demands put on the construction of theories and by so doing it speeds up the process of delivering tools for the development of technologies, at the price of sacrificing understanding.

Pilot waves. The instrumentalist approach of the Copenhagen School is further developed in other interpretations of quantum mechanics. For example, the de Broglie–Bohm–Vigier approach [Vigier et al., 2000] wants to save not only Schrödinger's equations, special relativity and the point particle description but also the statistical reading, while preserving classical determinism. In pursuit of such a goal it introduces a "subquantal medium" in which particles move like pollen on the surface of a liquid. The analogy brings us back to an ether that in this case is stochastic. The example shows how fantasy is the price to be paid in this epistemological approach.

5.5.1. The present approach. In the presentation of relational electromagnetism (see Chapter 4) Maxwell equations represent the propagation of EM effects in regions outside its production sources. The sources of EM action are described only as Lagrangian parameters that can be identified as charge densities and current densities and satisfy the continuity equation. However, the entity of such densities inside what we intuitively call matter is not prescribed by electromagnetism. Extending EM into the material region promises then to produce the laws of QM.

We advance a unified action integral inspired in de Broglie and Schrödinger's intuitions as well as in Maxwell's electrodynamics (Lemma 5.1 to Theorem 5.1). Hamilton's variational principle appears as central to the development, since it separates circumstantial issues from "laws of nature". The law is what remains invariant regardless of initial or final conditions (or condition at any time)[67]. Hamilton's principle was born as cognitive surpass and evolved as a guide for the construction of physics, a history that situates it as the primary choice for an attempt to unify theories. Poincaré [Poincaré, 1913d] considered it a fundamental principle as well.

The classical laws of motion, Schrödinger's equation and the laws of electromagnetism all derive from the same principle of least action. It is proper to say that they form a unit of knowledge. There is no much surprise in this fact, the principle of minimal action was introduced by Hamilton as form of collecting the system of ordinary equations of Lagrangian mechanics in one equation in partial derivatives. In turn Lagrange introduced the form today known as Lagrangian in order to incorporate in one expression those force laws given by their effects (usually called constraints of motion) and those given as interactions (the original realm of Newton's theory).

In this endeavour we can continue to use the name "particle", but understanding a quantum particle as an unitary entity characterised by mass (which is the form of influencing through gravitational actions), charge, current (charge in motion) and magnetisation (spin), the latter ones being the attributes related to EM effects. Furthermore, we consider charge- and mass-densities to be proportional and currents to bear the same proportion with momentum than the charge-to-mass relation. These densities cannot be thought of by analogy with the densities resulting from aggregates of matter since it is not possible to have a self-action or an interaction with only a portion of the quantum particle[68].

A particle is indissolubly associated to its localisation when it is though as: "the particle is that thing that lies there". In such regard, the vibration of Faraday's rays[p. 447, Faraday, 1855] becomes dissociated from the particle. It is not an action performed by nature but rather by our way of knowing. In contrast, the "problem" of measurement in QM is dissolved in our construction, actually, all what is needed is a quantum mechanical description of the measurement device and its (usually) electromagnetic form of interacting with the system of interest.

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[67] The original concept of variation by Hamilton reads "...any arbitrary increment whatever, when we pass in thought from a system moving in one way, to the same system moving in another, with the same dynamical relations between the accelerations and positions of its points, but with different initial data..."[Hamilton, 1834].

[68] Readers familiar with SR may claim that there is a requirement of simultaneity at different points that might violate postulates of SR. However, SR displays self contradictions [Solari and Natiello, 2022a] prior to any claim. Its logical problems are usually made invisible by the instrumentalist approach that persuades us not to examine the correspondence of the construction and the interpretation. Experimental results support our view as well, for example [Aspect et al., 1982].

The universe cannot be suppressed in an idealisation, its presence will be always in action, to the very least in the form of time. If in addition, we allow ourselves to separate the inferred (matter) from what is sensed (action) as it is done in the pair QM and EM, we simply create a fantasy, an illusion, where poor Schrödinger's cat is both dead and alive until we measure. Under the present view, the quantum mechanical entity will have associated time-dependent electromagnetic fields in as much it has not reached a stationary state. A state initially described as a superposition of stationary states will evolve into an apparent stationary state if we agree to ignore the relation with the electromagnetic energy whose space locus has no longer an overlap with the space locus of the material entity. Notice that in order to make this statement we have to include the localisation between the attributes of the entity creating a second entity, the photon, associated to the EM radiated energy.

The wave function in our approach is associated to the distribution of charge, currents and magnetisation. The need to use a wave function and not the densities directly is taken from de Broglie's argumentation to make contact with experiments that relate frequency to energy involving Planck's constant. We must highlight that when analysing a free particle, the kinetic energy appears as the result of the above mentioned hypothesis and the conservation of charge, resulting in the usual $\frac{p^2}{2m}$, not as analogy or correspondence with classical mechanics but rather to be consistent with electromagnetism.

The wave function is an attribute that keeps track of the state of the particle and it is used primarily to determine the interaction (electromagnetic or gravitational) with other systems. In turn, the state of the particle is not just an attribute of itself but of its environment as well. Such view may result odd to physicists and chemists while it is the standard view in ecology. Thus, the problem of measurement is incorporated from the beginning since measuring means to enter in contact with a device.

Cognitive surpass means in this regard that the natural laws that govern the interaction of electron and proton as spatially separated entities must be the same as those governing their microscopic interaction, the differences corresponding to the context. When the electron and proton conform a new entity, the hydrogen atom, they act as a unity and interact with the environment as an atom. This observation suggests that the atomic wave function may be factored as $|\Psi\rangle = |\psi_r\rangle|\psi_{cm}\rangle$ while when we consider electron and proton as not entangled, it factors as $|\Psi\rangle = |\psi_p\rangle|\psi_e\rangle$, these two expressions constituting two different limits of the total wave function. This simple argument allows us to construct, the Hamiltonian of the hydrogen atom without resource to analogies, empirical corrections such as gyromagnetic factors or "relativistic corrections", i.e., patches inspired by SR needed to match experimental results.

The quantum mechanics that emerges from electromagnetism has been tested in the construction of the internal energies for the Hydrogen atom producing outstanding results, superior to those of standard quantum mechanics. Equal in precision but definitely superior in consistence and consilience. Another insight gained is that the moving spin reveals its role as the carrier of *electric* polarisation in the spin-orbit interaction.

Our view is incompatible with the statistical interpretation of QM where measuring is a sort of magic act in which the laws of QM do not apply. As Carroll puts it "quantum mechanics, as it is enshrined in textbooks, seems to require separate rules for how quantum objects behave when we're not looking at them, and how they behave when they are being observed" [Carroll, 2019]. Furthermore, a condition for a theory to be considered rational consists in the possibility of "filling the gap", which means that the details left without specification can be provided in accordance to the theory. The current QM is unable to specify the details of the measurement, they remain in an abstract sphere where apparently, for example, the position of the electron relative to the proton in the hydrogen atom can be determined with as much precision as desired, yet not a word is said on how such measurement would be performed. It fails then a requisite of rationality as discussed in general form in Chapter 6.

The consilience of the present formulation not only contrasts with the statistical interpretation of quantum mechanics, it contrasts with what is implied in the notion of interpretation, i.e., that equations are first herded together as needed in terms of their usefulness without regard to their meaning, and a posteriori meaning is attributed in the "interpretation" to promote acceptance and facilitate the use of the equations. Such procedure is not needed, it is not forced by the limitations of our mind and much less by Nature. Any claim regarding the Truth in such collections is ludicrous. However, students in sciences are indoctrinated constantly in terms of the Truth value of such "theories". It is the social power of the guild of physicists what prevents progress towards understanding.

5.5.2. Obstacles for the unification of physics. There are two obstacles that have to be overcome to grasp the unification of physics enacted in our presentation. The first is the entrenched belief in the representation of fundamental quantum particles as points. Schrödinger was, in our limited knowledge, the only one to point out this problem, yet he did not notice that this view was threaded with special relativity.

The usual form used to incorporate the Lorentz force in the case of point particles was the constructive basis for several authors such as Pauli [Pauli Jr, 1927] and Dirac [Dirac, 1928]. This led to the identification of the momentum p with the operator $-i\hbar\nabla$ (in all situations), a compensating error sustained in Bohr's principle of correspondence. In turn, this chain of decisions made impossible to associate the operator with the current and prevented the production of EM interactions in QM except by resource to analogy.

In contrast, in the present approach the unity of concepts is sought in the fundamentals. The Lorentz force was integrated to Maxwell equations in Subsection 4.3.5, without restrictions to point particles, using Lorentz' Lagrangian formulation. It is worth mentioning that so far there has been no need to request the quantification of the EM field. All results are consistent with Planck's view [Brush, 2002] who recognised that the exchange of energy in absorption and emission processes was quantified yet, this fact does not imply that the field itself needs to be quantified a priori, being the latter an idea usually attributed to Einstein [Einstein, 1905b] (cf. Subsection 5.4.7).

The second obstacle is the belief that special relativity is the cornerstone of electromagnetism.

Einstein's instrumentalism is declared in:

> Physics constitutes a logical system of thought which is in a state of evolution, and whose basis cannot be obtained through distillation by any inductive method from the experiences lived through, but which can only be attained by free invention. The justification (truth content) of the system rests in the proof of usefulness of the resulting theorems on the basis of sense experiences, where the relations of the latter to the former can only be comprehended intuitively. [Einstein, 1936, p. 381]

SR closed the debate around the ether at the price of introducing elitism [Essen, 1978, Lerner, 2010], and dropping the advances made by Lorentz in terms of a Lagrangian formulation of Maxwell's theory. The problem is not Einstein, all of us are limited and to show him limited only proves what is not disputed: Einstein was human. The force driving the social construction of idols is the profit obtained by the clergy. In situations like this, the philosophical debate is obscured and impeded by the social struggle. Any one who has objected the idea of SR as the cornerstone of EM has been threatened by the guild of physics. We came to know only the testimony of those strong enough to (socially) survive the attacks. Let us name them: Essen [Essen, 1971, 1978] who wrote as abstract of his 1978 intervention that "The theory is so rigidly held that young scientists dare not openly express their doubts". Essen provided evidence that SR does not make sense in experimental terms. Dingle, warned us about the epistemological problems of SR [Dingle, 1960a], the relevance of the Doppler effect by itself [Dingle, 1960b] and gave testimony of the attacks he endured [Dingle, 1972]. Phipps insisted from both theoretical and experimental points of view in the relational view of physics [Phipps Jr, 2006, Phipps, 2014].

If we go back in time, the men in charge of producing theories about nature were philosophers-mathematicians. For Galileo, mathematics provided the language to understand the universe by the philosopher, who was understood as a master at reasoning. When the production of scientific results was "industrialised" a new social character was born: the scientist (in particular, the physicist). The professional scientist, produced at the universities, was declared free from the control of philosophy (i.e., reason), each science had its method and forms of validations, it was proclaimed. The guild of the profession was then left in charge of the quality control. The immediate consequence of this social change in physics was a new form of explanation that left behind natural philosophy and empowered the "second physicist" [Jungnickel and McCormmach, 2017] with the social responsibility of producing theories. The technical success of electromagnetism contrasted with the failure of the new epistemology that had made of the ether the weapon for attacking the remains of the old epistemology, for example Göttingen's electromagnetism (see Chapter 4). When it became clear that the ether was a no-end route, the task of saving the theoretical physicist became urgent. The merit of SR is this social rescue and it is also the ultimate reason for not being challenged.

Appendix

5.A. Proof of variational results

5.A.1. Proof of Lemma 5.1:

Proof. Since the wave function admits the independent variations $\psi \mapsto \psi + \delta\psi$ and $\psi \mapsto \psi + i\delta\psi$ we have that

$$\begin{aligned} \frac{\delta\mathcal{A}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{A}}{\delta\psi^*}\delta\psi^* &= 0 \\ \frac{\delta\mathcal{A}}{\delta\psi}(i\delta\psi) + \frac{\delta\mathcal{A}}{\delta\psi^*}(-i\delta\psi^*) &= 0 \end{aligned}$$

where $\delta\psi^* = (\delta\psi)^*$, and hence, it is sufficient to consider the variations of ψ and ψ^* as independent variations that must result in a null change of $\mathcal{A}$. By partial integration we obtain $-i\hbar\dot{\psi}^*\psi = -i\hbar\left(\psi^*\psi\right)_{,t} + i\hbar\psi^*\dot{\psi}$ and the stationary action corresponds to

$$\delta\mathcal{A}_{QM} = \int_{t_0}^{t_1} ds \left\langle \delta\psi^* \left[i\hbar\dot{\psi} + \frac{\hbar^2}{2m}\Delta\psi \right] \right\rangle = 0.$$

□

5.A.2. Proof of Lemma 5.2.

Proof. We act the variations piecewise, recalling that $\delta\psi = \epsilon\mu$ gives $\delta\dot{\psi} = (\epsilon\mu)_{,t} = \dot{\epsilon}\mu + \epsilon\dot{\mu}$,

$$\begin{aligned} \mathcal{A}_1 &= \frac{i\hbar}{2}\int_{t_0}^{t_1} ds \left\langle \psi^*\dot{\psi} - \dot{\psi}^*\psi \right\rangle \\ \delta\mathcal{A}_1 &= \frac{i\hbar}{2}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\dot{\psi} - \dot{\psi}^*\mu \right\rangle + \frac{i\hbar}{2}\int_{t_0}^{t_1} ds \left\langle \psi^*\left(\epsilon\mu\right)_{,t} - \left(\epsilon\mu\right)^*_{,t}\psi \right\rangle \\ &= \frac{i\hbar}{2}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\dot{\psi} - \dot{\psi}^*\mu \right\rangle - \frac{i\hbar}{2}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \dot{\psi}^*\mu - \mu^*\dot{\psi} \right\rangle \\ &= \int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^*\mu \right\rangle \end{aligned}$$

after a partial integration in time of the $\dot{\epsilon}$ contribution (in the second line) that does not modify the variation. Partial integrations in space using Gauss' theorem give

$$\begin{aligned}\mathcal{A}_2 &= \frac{\hbar^2}{2m}\int_{t_0}^{t_1} ds\,\langle\psi^*\Delta\psi\rangle\\ \delta\mathcal{A}_2 &= \frac{\hbar^2}{2m}\int_{t_0}^{t_1}\epsilon\; ds\,\langle\mu^*\Delta\psi+\psi^*\Delta\mu\rangle\\ &= \frac{\hbar^2}{2m}\int_{t_0}^{t_1}\epsilon\; ds\,\langle\mu^*\Delta\psi+\mu\Delta\psi^*\rangle\,.\end{aligned}$$

Since the variation complies with Maxwell equations, after some partial integrations, we have:

$$\begin{aligned}\delta\frac{1}{2}\int_{t_0}^{t_1} ds\left\langle\frac{1}{\mu_0}B^2-\epsilon_0E^2\right\rangle &= \int_{t_0}^{t_1} ds\left\langle\frac{1}{\mu_0}\delta B_m\cdot B-\epsilon_0\delta E_m\cdot E\right\rangle\\ &= \int_{t_0}^{t_1} ds\left\langle\frac{1}{\mu_0}A\cdot(\nabla\times\delta B_m)-V\delta(\epsilon_0\nabla\cdot E_m)-\epsilon_0 A\cdot\delta E_{m,t}\right\rangle\\ &= \int_{t_0}^{t_1} ds\,\langle A\cdot\delta j_m-V\cdot\delta\rho_m\rangle\end{aligned}$$

where $\delta j_m=\epsilon\mathfrak{J}_1+\dot{\epsilon}\mathfrak{J}_2$ and $\delta\rho_m=\epsilon\mathfrak{X}$. Consequently,

$$\begin{aligned}\delta\frac{1}{2}\int_{t_0}^{t_1} ds\left\langle\frac{1}{\mu_0}B^2-\epsilon_0E^2\right\rangle &= \int_{t_0}^{t_1} ds\,\langle A\cdot\epsilon\mathfrak{J}_1+\dot{\epsilon}A\cdot\mathfrak{J}_2-V\epsilon\mathfrak{X}\rangle\\ &= \int_{t_0}^{t_1}\epsilon\; ds\left\langle A\cdot\mathfrak{J}_1-\left(A\cdot\mathfrak{J}_2\right)_{,t}-V\mathfrak{X}\right\rangle\end{aligned}$$

after another partial integration in time. □

5.A.3. Proof of Corollary 5.1.

Proof. $\mathfrak{J}_2=-\epsilon_0\mathfrak{L}$ satisfies the assumption $\mathfrak{X}+\nabla\cdot\mathfrak{J}_2=0$ of Lemma 5.2, as well as $\mathfrak{J}_2+\nabla\times W$ does it, for any suitable W. However, setting $\mathfrak{J}_1:=\mathfrak{J}_1+\nabla\times W_{,t}$, the vector W does not appear in any of the equations of the Lemma. Note that the precise form of $\mathfrak{J}_1$ as a function of μ has not been prescribed so far. □

5.A.4. Proof of Theorem 5.1.

Proof. Note that the column $\mathfrak{X}$ in the table corresponds to propagating $\delta\psi$ on the quantum charge density $\rho=\psi^*\psi$.

For the "time/energy" entry, the variation of the action corresponds to setting $\mu\equiv\dot{\psi}$ in Lemma 5.2. Since ϵ is any possible variation function, the only way to ensure a stationary action is to let the space integral be identically zero, namely:

$$\left\langle\mu^*\left(i\hbar\dot{\psi}\right)+\left(i\hbar\dot{\psi}\right)^*\mu\right\rangle+\frac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi+\mu\Delta\psi^*\right\rangle+\left\langle A\cdot\mathfrak{J}_1-\left(A\cdot\mathfrak{J}_2\right)_{,t}-qV\mathfrak{X}\right\rangle=0.$$

It follows that $\dot{\psi}^*\left(i\hbar\dot{\psi}\right)+\left(i\hbar\dot{\psi}\right)^*\dot{\psi}=0$ and $\dot{\psi}^*\Delta\psi+\dot{\psi}\Delta\psi^*=(\psi^*\Delta\psi)_{,t}$, by repeated application of Gauss' theorem. Letting $\mathfrak{J}_1=-2\epsilon_0\mathfrak{L}_{,t}-\mathfrak{j}$, we propose

$\mathfrak{j} = j_{,t}$ so that $\mathfrak{X}_{,t} + \nabla \cdot \mathfrak{j} = 0$. Since $\mathfrak{X} = (\psi^*\psi)_{,t}$, we have that $\mathfrak{J}_2 = -\epsilon_0 \mathfrak{E} = j$ and $\mathfrak{J}_1 = j_{,t}$. The conditions of Lemma 5.2 are satisfied and

$$\begin{aligned} A \cdot \delta j - V \delta\rho &= \epsilon \left[A \cdot \mathfrak{J}_1 - \left(A \cdot \mathfrak{J}_2 \right)_{,t} - V \mathfrak{X} \right] \\ &= \epsilon \left[A \cdot \left(\mathfrak{J}_1 - \mathfrak{J}_{2,t} \right) - A_{,t} \cdot \epsilon_0 \mathfrak{E} - V \mathfrak{X} \right] \\ &= -\epsilon \left[- \left(A_{,t} + \nabla V \right) \cdot \epsilon_0 \mathfrak{E} \right] \\ &= -\epsilon \left[E \cdot j \right] \end{aligned}$$

The latter expression is the time-derivative of the electromagnetic energy:

$$\begin{aligned} \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2)_{,t} &= E \cdot \epsilon_0 E_{,t} + \frac{1}{\mu_0} B \cdot B_{,t} \\ &= E \cdot \left(\frac{1}{\mu_0} \nabla \times B - j \right) - \frac{1}{\mu_0} B \cdot (\nabla \times E) \\ &= -E \cdot j + \frac{1}{\mu_0} \nabla \cdot (B \times E) \end{aligned}$$

By Gauss' theorem, the divergence vanishes when integrated and we finally obtain the equation

$$\left\langle \partial_t \left[-\frac{\hbar^2}{2m} \left(\psi^* \Delta \psi \right) + \frac{1}{2} \left(\frac{1}{\mu_0} B^2 + \epsilon_0 E^2 \right) \right] \right\rangle = 0$$

expressing the conservation of energy.

For the "position/velocity" entry in the table, in order to obtain a stationary action integral, a similar argument about the independence and arbitrariness of the real-valued time function ϵ forces the following vector valued integral to be zero:

$$\begin{aligned} \left\langle -i\psi^* \left(k \cdot x \right) \left(i\hbar\dot{\psi} \right) + \left(i\hbar\dot{\psi} \right)^* i \left(k \cdot x \right) \psi \right\rangle &+ \frac{\hbar^2}{2m} \left\langle -i\psi^* \left(k \cdot x \right) \Delta\psi + i \left(k \cdot x \right) \psi \Delta \psi^* \right\rangle \\ &+ \left\langle A \cdot \mathfrak{J}_1 + \left(A \cdot \mathfrak{E} \epsilon_0 \right)_{,t} - V \mathfrak{X} \right\rangle \end{aligned}$$

where $\mathfrak{X} = (ix\psi)^*\psi + \psi^*(ix\psi) = 0$. Hence, we have $-\mathfrak{J}_2 = \epsilon_0 \mathfrak{E} = 0$. The continuity equation is satisfied by taking $\mathfrak{J}_1 = 0$. The electromagnetic integral is identically zero and we obtain

$$\left\langle k \cdot \left(-i\psi^* x \left(i\hbar\dot{\psi} \right) + \left(i\hbar\dot{\psi} \right)^* ix\psi \right) \right\rangle + \frac{\hbar^2}{2m} \left\langle k \cdot \left(-i\psi^* x \Delta\psi + ix\psi\Delta\psi^* \right) \right\rangle = 0.$$

The first term gives $k \cdot \left(\hbar \partial_t \left\langle \psi^* x \psi \right\rangle \right)$ while the second reduces to $\frac{i\hbar^2}{2m} k \cdot \left\langle \psi^* \left(\nabla\psi \right) - \left(\nabla\psi \right)^* \psi \right\rangle$ after repeated application of Gauss' theorem. Finally,

$$0 = \partial_t \left\langle \psi^* x \psi \right\rangle + \frac{i\hbar}{2m} \left\langle \psi^* \left(\nabla\psi \right) - \left(\nabla\psi \right)^* \psi \right\rangle = \partial_t \left\langle \psi^* x \psi \right\rangle - \frac{1}{q} \langle j \rangle.$$

For the "displacement/momentum" case, we compute $\mathfrak{X} = r \cdot \nabla\rho$, set $\mathfrak{j} = (r \cdot \nabla)\, j$ and $\mathfrak{J}_2 = -\epsilon_0 \mathfrak{E} = -r\rho$. Then $\mathfrak{J}_1 - \mathfrak{J}_{2,t} = \epsilon_0 \mathfrak{E}_{,t} + \mathfrak{j}$, which computes to

$\mathfrak{I}_1 - \mathfrak{I}_{2,t} = r\rho_{,t} + (r\cdot\nabla)j = -r(\nabla\cdot j) + (r\cdot\nabla)j$ and further $-r(\nabla\cdot j)+(r\cdot\nabla)j = -\nabla\times(r\times j)$. Hence, up to a global divergence vanishing by Gauss' theorem,

$$\begin{aligned} A\cdot\left(\mathfrak{I}_1 - \mathfrak{I}_{2,t}\right) - A_t\cdot\mathfrak{I}_2 - V\mathfrak{X} &= A\cdot(-\nabla\times(r\times j)) + (A_t+\nabla V)\cdot(r\rho) \\ &= -A\cdot\nabla\times(r\times j) - E\cdot(r\rho) \\ &= -(r\times j)\cdot(\nabla\times A) - r\cdot\rho E \\ &= -r\cdot(\rho E + j\times B)\,. \end{aligned}$$

Finally, the arbitrariness of ϵ demands

$$r\cdot\left\langle(\nabla\psi)^*\left(i\hbar\dot\psi\right)+\left(i\hbar\dot\psi\right)^*\nabla\psi\right\rangle+\frac{\hbar^2}{2m}r\cdot\left\langle(\nabla\psi^*)\,\Delta\psi+(\nabla\psi)\,\Delta\psi^*\right\rangle - r\cdot\langle\rho E + j\times B\rangle = 0.$$

By repeated use of Gauss' theorem we obtain $\left\langle(r\cdot\nabla\psi)^*\,\Delta\psi+(r\cdot\nabla\psi)\,\Delta\psi^*\right\rangle = 0$. Since the directions r are arbitrary, we have

$$i\hbar\left\langle(\nabla\psi)^*\,\dot\psi-\dot\psi^*\nabla\psi\right\rangle = \frac{m}{q}\partial_t\,\langle j\rangle = \langle(j\times B)+\rho E\rangle\,.$$

For the "rotation/angular momentum" entry $\mathfrak{X} = \Omega\cdot(x\times\nabla\rho)$, $\mathfrak{I}_1 = \mathfrak{j} = (\Omega\cdot(x\times\nabla))\,j - \Omega\times j$ and we set $\mathfrak{I}_2 = -(\Omega\times x\rho)$ since $\nabla\cdot\mathfrak{I}_2 = -\Omega\cdot(x\times\nabla\rho) = -\nabla\cdot(\Omega\times x\rho)$.

The proposed variation $\delta\psi$ enters in the space integrals, which specifically become

$$\Omega\cdot\left\langle i\hbar\left((x\times\nabla\psi^*)\,\dot\psi-(x\times\nabla\psi)\,\dot\psi^*\right)+\frac{\hbar^2}{2m}\left((x\times\nabla\psi^*)\,\Delta\psi+(x\times\nabla\psi)\,\Delta\psi^*\right)\right\rangle.$$

The second term vanishes as in the previous case, while the first term reads

$$\Omega\cdot\left\langle x\times\left(i\hbar\left\langle(\nabla\psi)^*\,\dot\psi-\dot\psi^*\nabla\psi\right\rangle\right)\right\rangle \;=\; \frac{m}{q}\Omega\cdot\partial_t\,\langle x\times j\rangle$$

The electromagnetic part reads

$$\begin{aligned} A\cdot\left(\mathfrak{I}_1 - \mathfrak{I}_{2,t}\right) - A_t\cdot\mathfrak{I}_2 - V\mathfrak{X} &= A\cdot[\Omega\cdot(x\times\nabla)]\,j - \Omega\cdot(j\times A) \\ &+ \Omega\cdot((x\rho_{,t})\times A) - \Omega\cdot[(x\rho)\times E] \\ &= -\Omega\cdot(x\times(j\times B)) - \Omega\cdot[x\times\rho E] \\ &= -\Omega\cdot(x\times(\rho E + j\times B))\,. \end{aligned}$$

Being Ω a vector of arbitrary direction, equating the variation to zero we get

$$\frac{m}{q}\partial_t\,\langle x\times j\rangle - \langle x\times(\rho E + j\times B)\rangle = 0$$

which is the evolution of the angular momentum under the action of the Lorentz force. □

5.A.5. Proof of Lemma 5.3.

PROOF. We only need to check that $(\delta\rho)_{,t} + \nabla\cdot\delta j = 0$ which follows from computing

$$\mathfrak{I}_1 - \mathfrak{I}_{2,t} = -\epsilon_0\mathfrak{E}_{,t} - \mathfrak{j} = -\frac{1}{\mu_0}\nabla\times\mathfrak{B}$$

according to Maxwell's equations. Substituting in the result of the preparatory lemma and using the arbitrary $\epsilon(t)$ argument, the results follow. □

5.A.6. Proof of Lemma 5.4.

PROOF. We begin from eq. (5.2.5) of Lemma 5.3. We write $\mathfrak{X} = (\mu^*\psi + \mu\psi^*)$ and set $\mathfrak{j} = -\frac{i\hbar q}{2m}\left(\mu^*\nabla\psi - \psi\nabla\mu^* + \psi^*\nabla\mu - \nabla\psi^*\mu\right)$. According to Lemma 5.2 the variation of the electromagnetic contribution to the action reads:

$$\int_{t_0}^{t_1} \epsilon\, ds \left\langle A\cdot\mathfrak{J}_1 - \left(A\cdot\mathfrak{J}_2\right)_{,t} - V\mathfrak{X}\right\rangle$$

which according to Lemma 5.3 is:

$$\int_{t_0}^{t_1} \epsilon\, ds \left\langle -\frac{1}{\mu_0}A\cdot(\nabla\times\mathfrak{B}) - \epsilon_0 E\cdot\mathfrak{E}\right\rangle$$

The variation of the action becomes:

$$\int (\epsilon(s)\, ds)\left(\left\langle\mu^*\left(i\hbar\dot\psi\right) + \left(i\hbar\dot\psi\right)^*\mu\right\rangle + \frac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi + \mu\Delta\psi^*\right\rangle - \left\langle -\frac{1}{\mu_0}A\cdot\left(\nabla\times\mathfrak{B}\right) - \epsilon_0 E\cdot\mathfrak{E}\right\rangle\right).$$

Letting $\epsilon(t)$ to be constant, integrating in $-T \le s \le T$, and letting $T\to\infty$, projects μ into its ω Fourier component, μ_ω, while the electromagnetic terms integrate to $A\cdot\mathfrak{j}_\omega - V\mathfrak{X}_\omega$ projecting in the same form the variation of density and current, with $\mathfrak{X}_\omega = (\mu^*_\omega\psi_\omega + \mu_\omega\psi^*_\omega)$ and $\mathfrak{j}_\omega = -\frac{i\hbar q}{2m}\left(\mu^*_\omega\nabla\psi_\omega - \psi_\omega\nabla\mu^*_\omega + \psi^*_\omega\nabla\mu_\omega - \nabla\psi^*_\omega\mu_\omega\right)$. Since μ is arbitrary, its Fourier components are arbitrary and the null variation of the action results in the requisite:

$$\langle\mu^*_\omega\left(\hbar\omega\psi_\omega\right)\rangle + \frac{\hbar^2}{2m}\langle\mu^*_\omega\Delta\psi_\omega\rangle + \left\langle A\cdot\left(\frac{i\hbar q}{2m}\right)\left(\mu^*_\omega\nabla\psi_\omega - \psi_\omega\nabla\mu^*_\omega\right) - V\nu^*_\omega\psi_\omega\right\rangle = 0.$$

Partial integration gives $A\cdot(\nu^*\nabla\psi_\omega - \psi_\omega\nabla\nu^*) = A\cdot(2\nu^*\nabla\psi_\omega - \nabla\left(\psi_\omega\nu^*\right))$ and by Gauss' theorem $\langle -A\cdot\nabla\left(\nu^*\psi_w\right)\rangle = \langle\nu^*\psi_w\nabla\cdot A\rangle$. For an homogeneous magnetic field we have $A = \frac{1}{2}x\times B$ (and hence $\nabla\cdot A = 0$) and further $A\cdot\left(\frac{i\hbar q}{2m}\right)2\nu^*\nabla\psi_\omega = \nu^*\left(\frac{i\hbar q}{2m}\right)(x\times B)\cdot\nabla\psi_w$, which finally leads to

$$\left\langle\nu^*\left(\hbar\omega\psi_\omega + \frac{\hbar^2}{2m}\Delta\psi_\omega - \frac{i\hbar q}{2m}B\cdot\left(x\times\nabla\psi_\omega\right) - qV\psi_\omega\right)\right\rangle = 0.$$

It follows that $\hbar\omega\psi \equiv \mathcal{E}\psi = -\frac{\hbar^2}{2m}\Delta\psi - \frac{q}{2m}B\cdot\left(x\times\left(-i\hbar\nabla\right)\psi\right) + qV\psi.$ □

5.A.7. Proof of Lemma 5.5.

PROOF. Letting $\delta\psi = \epsilon\hat{O}\psi$, the operator $\hat{O}$ must be linear as a consequence of the infinitesimal character of variations. Acting with Θ on the continuity equation we get

$$\begin{aligned} \Theta\left(\rho_{,t}+\nabla\cdot j\right) &= 0 \\ \left(\Theta\rho\right)_{,t}+\Theta\left(\nabla\cdot j\right) &= 0 \end{aligned}$$

provided $\hat{O}_{,t}=0$. Since

$$\begin{aligned} \Theta\left(\nabla\cdot j\right) &= \nabla\cdot\left(\Theta j\right)-\frac{i\hbar q}{2m}\left(\psi^*\left(\left[\nabla,\hat{O}\right]\cdot\nabla\right)\psi-\psi\left(\left[\nabla,\hat{O}\right]\cdot\nabla\psi\right)^*\right) \\ &\quad -\frac{i\hbar q}{2m}\left(\left(\left[\nabla,\hat{O}\right]\psi\right)^*\cdot\nabla\psi-\nabla\psi^*\cdot\left(\left[\nabla,\hat{O}\right]\psi\right)\right) \end{aligned}$$

It is then possible to propose a suitable current $\mathfrak{j}$ if and only if

$$-\frac{i\hbar q}{2m}\left(\psi^*\left(\left[\nabla,\hat{O}\right]\cdot\nabla\right)\psi-\psi\left(\left[\nabla,\hat{O}\right]\cdot\nabla\psi\right)^*+\left(\left[\nabla,\hat{O}\right]\psi\right)^*\cdot\nabla\psi-\nabla\psi^*\cdot\left(\left[\nabla,\hat{O}\right]\psi\right)\right)=\nabla\cdot K \tag{5.A.1}$$

for some vector K. In such a case, $\mathfrak{j}=\Theta j+K$, satisfies $\left(\Theta\rho\right)_{,t}+\nabla\cdot\mathfrak{j}=0$.

Let us further investigate condition eq. (5.A.1). Clearly, it implies

$$-\frac{i\hbar q}{2m}\left\langle\left(\psi^*\left(\left[\nabla,\hat{O}\right]\cdot\nabla\right)\psi-\psi\left(\left[\nabla,\hat{O}\right]\cdot\nabla\psi\right)^*+\left(\left[\nabla,\hat{O}\right]\psi\right)^*\cdot\nabla\psi-\nabla\psi^*\cdot\left(\left[\nabla,\hat{O}\right]\psi\right)\right)\right\rangle=\left\langle\nabla\cdot K\right\rangle$$

Introducing the Hermitian adjoint, the lhs can be rewritten as follows:

$$\frac{i\hbar q}{2m}\left\langle\psi^*\left(\left[\Delta,\hat{O}\right]-\left[\Delta,\hat{O}\right]^\dagger\right)\psi\right\rangle = \left\langle\nabla\cdot K\right\rangle.$$

Hence, if $\left[\Delta,\hat{O}\right]$ is Hermitian (or equivalently if $\hat{O}^\dagger=-\hat{O}$, since Δ is itself Hermitian), eq. (5.A.1) ensures that an adequate K exists with $\left\langle\nabla\cdot K\right\rangle=0$. Conversely, if a vector field K exists such that

$$\int d^3x\ \left(\nabla\cdot K\right)=\int_\Sigma K\cdot\widehat{d\Sigma}=0$$

where the volume integral is taken as limit over all space and Σ is the oriented surface bounding the volume of integration, then $\hat{O}$ is anti-Hermitian. □

5.B. Scholium. Instrumentalism and Dirac's electron model.

5.B.1. Instrumentalist epistemology revisited. Boltzmann's appraisal of Hertz' approach (see quotation by Boltzmann in Section 3.1) addresses a central issue in technology development and its accompanying economic profit. The idea is that achieving a functioning path to the desired goal (be it developmental or economic) is what matters the most and what ultimately justifies the procedure is the successful final result. Hence the need of enlarging the basis of well trained technicians by the introduction of a "second physicist" (cf. Section 3.2).

It must be emphasised that the problem to be faced is not the construction of knowledge, which was the problem solved for electromagnetism by Maxwell, but rather the *student's problem*: How do we accept this formulae? We emphasise the word acceptance which means a general agreement that the received formulae are satisfactory.

Boltzmann's answer corresponds to the student's question. In the first place, it must be indicated that acceptance of given instruction is useful to pursue a professional education. The student is in a conflict of interests, they do not put themselves in the place of the constructor, they do not intend to be a Maxwell, only to have the usufruct of Maxwell's ideas. Hertz confesses he does not understand Maxwell[Hertz, 1893, Introduction] and by this he means that he cannot grasp the legitimacy of the construction, its reasonability, much less Maxwell doubting about the ether, for example. Hertz decides to drop the rational construction while keeping the equations. In so doing, the student misses every opportunity of developing autonomous and critical thought. These students however accept the argument of requiring some "formal conditions" they have learnt from teaching habits but they do not know these conditions from reason. After Hertz, a frequent attitude in Physics has been to "find the equation", giving little or no attention to the underlying physical phenomenon (e.g., Bohr's atomic model or the quest for Dirac's equation).

Reason is mostly absent in this picture and it is absent as well in the alternative "phenomenology" mentioned. As for Boltzmann relying onto experience, Goethe has prevented us in his aphorism #1072 stating that experiential knowledge is only half of experience (see Foreword).

Mach's writings help us to understand the teacher's side of the problem.

> Man acquires his first knowledge of nature half-consciously and automatically, from an instinctive habit of mimicking and forecasting facts in thought, of supplementing sluggish experience with the swift wings of thought, at first only for his material welfare. When he hears a noise in the underbrush he constructs there, just as the animal does, the enemy which he fears; when he sees a certain rind he forms mentally the image of the fruit which he is in search of; just as we mentally associate a certain kind of matter with a certain line in the spectrum or an electric spark with the friction of a piece of glass [...]
>
> The **first real beginnings of science** appear in society, particularly in the manual arts, where the **necessity for the communication of experience** arises. Here, where some new discovery is to be described and related, the compulsion is first felt of clearly defining in consciousness the important and essential features of that discovery, as many writers can testify. The aim of instruction is simply the saving of experience; the labor of one man is made to take the place of that of another [...]
>
> The communication of scientific knowledge always involves description, that is, a **mimetic reproduction of facts in thought, the object of which is to replace and save the trouble of new experience**. Again, **to save the labor of instruction and of acquisition, concise, abridged description is sought**. This is really all that natural laws are. Knowing the value of the acceleration of gravity, and Galileo's laws of descent, we possess simple and compendious directions

> for reproducing in thought all possible motions of falling bodies. A formula of this kind is a complete substitute for a full table of motions of descent, because by means of the formula the data of such a table can be easily constructed at a moment's notice without the least burdening of the memory.
>
> No human mind could comprehend all the individual cases of refraction. But knowing the index of refraction for the two media presented, and the familiar law of the sines, we can easily reproduce or fill out in thought every conceivable case of refraction. The advantage here consists in the disburdening of the memory; an end immensely furthered by the written preservation of the natural constants. More than this comprehensive and condensed report about facts is not contained in a natural law of this sort. In reality, the law always contains less than the fact itself, because it does not reproduce the fact as a whole but only in that aspect of it which is important for us, the rest being either intentionally or from necessity omitted. Natural laws may be likened to intellectual type of a higher order, partly movable, partly stereotyped, which last on new editions of experience may become downright impediments. Mach [2012, The economical nature of physical inquiry](Emphasis added)

Mach's account of the primitive form of acquiring knowledge could only appear as correct to those that have never confronted the phenomena and had to create some kind of knowledge. For Mach there are "facts", he fails to realise what Goethe intended to teach us with his aphorism #575 (see Foreword).

Mach's view discloses a serious lack of self-consciousness that is expected to make difficult, if not impossible, critical thinking. He cannot conceive science without communication, hence science becomes a social activity rather than a direct relation with the phenomena. Knowledge appears as a relation that can be communicated, and then language is performative[69] for science. Furthermore, imagination (fantasy) is associated by Mach to the abridged teaching, mind constructions replace observations. The view of the Enlightenment, a view that rejected in advance Mach's position in the words of Wilhelm von Humboldt quoted in Subsection 1.A.1: such instrumentalist attitude degrades human beings.

Mach and later Boltzmann and the Vienna Circle respond to a change of era, the end of the Enlightenment and the upcoming of the time of the masses [Ortega y Gasset, 1930].

5.B.2. Dirac's electron and hydrogen atom. We discuss here Dirac's presentation of his "relativistic hydrogen atom" to illustrate the instrumentalist approach, as opposed to the construction in this book. Dirac's approach has been multiply reproduced in textbooks and lecture notes over the years. It is remarkable

[69]"Performativity is the power of language to effect change in the world: language does not simply describe the world but may instead (or also) function as a form of social action." Performativity by J.R. Cavanaugh (link consulted on May 12, 2025).

that the followers[70] recognise that some consequences of Dirac's approach lack an explanation (or interpretation) while others conflict with preexisting knowledge and still others are not amenable to observational confrontation. These singularities are set aside or simply stated "for the record" without taking any action in view of the conflict. The focus stays on the portion of results that provide instrumentalist success.

Dirac's work [Dirac, 1928] is based upon a few premises:

(1) The forms correspond to Schrodinger's formulation of mechanics,
$$i\hbar\frac{\partial\psi}{\partial t} - H\psi = 0$$
being H the hamiltonian. Additionally, the momentum operator is $p = -i\hbar\nabla$.

(2) The Klein-Gordon equation is rejected because when conjugated it produces something that can be interpreted as an equivalent equation for a charge $-e$ (a *positron*, it would be said in current language) and negative energy. Dirac fears that arbitrary perturbations (which are implicitly considered representative of physics) would allow for transitions from positive to negative energy. However, using Born's interpretation of quantum mechanics he concludes that the equation allows for states mixing electron and positron solutions.

(3) The quantum equation of the electron must be *equivariant* with respect to Lorentz' transformations (classical formulations are invariant with respect to their transformation *groups*).

(4) The results must account for measured atomic energy differences (emission and absorption lines).

(5) In the case of the hydrogen atom, the theory considers an electron in the electrostatic environment of a point charge. No attention is given to the problems posed by point charges.

(6) In addition, the atomic nucleus has no associated spin or any other additional property.

Such construction follows the blueprints shown by Boltzmann and indulges itself in not giving reasons but just plausibility arguments that can encourage us to believe they are correct, i.e., that can convince us (especially for those that believe in special relativity, another theory constructed with similar blueprints). In his 1928 work he did not try to solve point 2. He addressed it only in [Dirac, 1934] by introducing a fantasy: an infinite sea of negative energy electrons[71]. It is worth noticing the difference between Pauli's construction of spinors and matrix operators elaborating from the experimental results of Uhlenbeck, Goudsmidt and Zeeman in order to adapt the model to the observations [Pauli Jr, 1927] with Dirac's attempt to force a linear PDE in order to adapt to previously existing equations.

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[70] see e.g. [Littlejohn, 2021a]and [Littlejohn, 2021b].

[71] The idea of an infinite aggregation of electrons does not survive scrutiny. The sea would have associated infinite mass and infinite charge which nevertheless would not produce any observable effects. The hypothesis is there to "fix" the issue of negative energies, but at the same time it destroys the connection with mechanics and electromagnetism.

A simple test can be run on Dirac's equation, namely how they account for Thomson's [Thomson, 1881] and Kaufmann's experiments [Kaufmann, 1906, Rogers et al., 1940] (see Appendix 4.C). According to Dirac his Hamiltonian is "real" (meaning in today's jargon: Hermitian) and the wave function is a 4-dimensional vector $|\psi\rangle$, hence

$$i\hbar\frac{d}{dt}\langle\psi|X|\psi\rangle = \langle\psi|\,[H,X]\,|\psi\rangle$$

for any operator X independent of time, such as for example the position operator. In Heisenberg's notation

$$i\hbar\frac{dX}{dt} = [H,X]\,.$$

According to Dirac's Hamiltonian, for X the position operator:

$$i\hbar\frac{dX}{dt} = [H,X] = i\hbar C\rho_1\sigma$$

being $\rho_1\sigma$ an array (vector) of three 4x4 matrices based upon the Pauli matrices, σ_i which are also written as $4x4$ matrices. Dirac's Hamiltonian reads[72].

$$H = eA_0 + C\rho_1\sum_{\mu=1}^{3}\left(\sigma^\mu\left(\mathbf{p}_\mu + \frac{e}{C}A_\mu\right)\right) + \rho_3 mC^2$$

The correspondence with classical mechanics that lies in the origin of QM is thus completely broken, sacrificed in the altar of linearity. There is no form for such velocity (the time-derivative of the position is named velocity) to account for a relation to the momentum p of any kind, much less to account for Kaufmann's experiment or Thomson's or any other for what matters. The time-derivative of the position operator is independent of time, position and momentum. In a similar way, when the Lorentz force is computed through Dirac's formalism [Littlejohn, 2021a, p. 9], the required velocity is now $C\rho_1\sigma$ contrary to the fact that such velocity is amenable to experimental determination, taking as many different values as there has been experiments over the years.

In short, the theory when applied to a wider collection of experiments than those used to persuade us, collapses completely. It also fails to connect to the starting point in the abduction of QM, which is classical mechanics.

The procedure followed by Dirac can be made explicit:

(1) Renounce to understand where negative energies had been introduced in the Klein-Gordon equation (an easy task).
(2) Require an equation invariant with respect to Lorentz' transformations. Choose it to be Schrodinger's equation, ignoring the existence of an invariant variational principle.
(3) Demand that the equation must match Schrödinger's equation in form (thus being first order in time) , not only in the limit $C \to \infty$ as a correspondence principle would require.
(4) Impose for the sake of linearity that the wave function must be a four-vector.

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[72]Indeed, this is explicitly computed in [Littlejohn, 2021a].

(5) Make no attempt at recovering the classical results for the equation of motion as demanded by Bohr's correspondence principle. Actually, as we have shown, it fails completely in this respect.
(6) Instead of recovering Lorentz' force for a pointlike particle as a limit, impose pointlike particles and a replacement formulation of equations of motion in the presence of Lorentz' forces, a substitute that works only for pointlike particles.

The list shows the actions that make the theory lose contact with previous theories such as electromagnetism and classical mechanics but it does not explain why it's social success. What the theory achieves is

(1) Show that the equations satisfy in some form a relativistic symmetry demanded a priori.
(2) Produce the same transition formulae than previous theories that had matched the experimental results but failed the previous requisites.

This is, the paper presents as a construction what really is a demonstration that the formulae in hand satisfies the two criteria required by instrumentalism.

It should be noted that Dirac's epistemological position moved away from instrumentalism in his late professional years. Kragh's account illustrates the transformation (see quotation in Subsection 3.4.1). By 1938 he had entered in a sort of crisis recognising that quantum electromagnetism (QED for quantum electrodynamics) required to accept things that even an instrumentalist as himself could not grant on the name of utility. The same happened to Feynman who rejected his creature QED [Feynman, 1983] (see Dirac's and Feynman's quotations towards the end of the Foreword).

The inconsistency conflict finally took the lead. It is reported that on another occasion, Dirac stated that:

> the agreement with observation is presumably a coincidence, just like the original calculation of the hydrogen spectrum with Bohr orbits [p. 347, note 80 Ch. 8, Kragh, 1990].

5.C. Scholium. Other paths to quantum mechanics

The paths to quantum mechanics are expected to be multiple, there is no reason to be otherwise, much less there is a reason to expect that the final equations, in as much as they can be contrasted to experiments, are going to be different. This is, matching the available experimental results does not select completely between theories. There is no reason for that and, as we have seen in Chapter 4, two different theories agree in all their formulae. This matter was highlighted by Maxwell as we have already indicated. However, the first satisfactory theory embraced has the advantage in front a new theory that offers to explain the same phenomena with same formulae. Where is the gain to made by the effort required for becoming acquainted with the new ideas? However, satisfaction is a subjective matter, even the satisfaction of a majority of scientists has lesser relevance than reason, so has Galileo taught us.

Let us tell a personal history that might be instructive.

When our journey began we believed in special relativity and the statistical interpretation of quantum theory, although we recognised that there were some rough parts as the measurement problem in quantum mechanics, the prohibition for the electron to radiate when being part of an atom, and several warnings with respect to SR that, unlike theories supported in reason, it appears to require instruction to the point of indoctrination. This indoctrination is manifest in the extreme homogeneity of the presentations of SR, most of them relying on a primitive, concrete, form of thinking with images, "clocks and rods". We though that a more rational presentation was possible for theories inherently correct. In this search we ended up constructing several theories that (apparently) match earlier attempts of conciliating electromagnetism and quantum theory. Let us show them.

Assume we do not know about Lorentz' introduction of his force and his electromagnetic Lagrangian. We may want to push a little bit forward the transversal and longitudinal masses as proposed by Lorentz, after all, they explain Kaufmann's experiment. If we write

$$\frac{dv}{dt} = F_{\|}/m_{\|} + F_{\perp}/m_{\perp}$$

where

$$\begin{aligned} m_{\|} &= m_0\gamma^3(v) \\ m_{\perp} &= m_0\gamma(v) \end{aligned}$$

and the term perpendicular and longitudinal refer to the direction of v, then we can construct a Lagrangian formulation using for the kinetic part of the Lagrangian

$$\mathcal{L}_K = -m_0 C^2\sqrt{1-(\frac{v}{C})^2}$$

(the "suitable relativistic Lagrangian" in [eq. 7.136 Goldstein, 1980]) with the immediate consequence that the momentum, p, reads

$$p = m_0\frac{v}{\sqrt{1-(\frac{v}{C})^2}} = \frac{\partial\mathcal{L}}{\partial v}.$$

An electromagnetic contribution of the form

$$\mathcal{L}_{EM} = q\,(v\cdot A - V)$$

(with the potentials given in the laboratory) produces the Lorentz force for a point particle.

The complete Lagrangian

$$\mathcal{L} = -m_0 C^2\sqrt{1-(\frac{v}{C})^2} + q\,(v\cdot A - V) \tag{5.C.1}$$

produces equations of motion that account for Kaufmann's results.

A problem appears when we try to conciliate this Lagrangian (and consequently the Hamiltonian) with the EM Lagrangian that helps us to understand the Doppler effect: they do not match, they differ in a Lorentz transformation. In the case of the Doppler effect we can consider how the EM waves produced in the laboratory are perceived by a receiver that moves with velocity u with respect to the laboratory. The fields, instead of acting on a pointlike particle act on a circuit, or at least a molecule. The field equations for the source can be obtained using eq. (4.3.10) varying it with respect to the potentials of the receiver. If the perceived field is to be obtained as a result, thus evidencing the Doppler effect, the fields for one of the systems (say the laboratory fields) have to be changed into the perceived fields using a Lorentz transformation (as in Chapter 4). The extreme simplicity of the pointlike particle hides these issues.

From the Lagrangian of eq. (5.C.1), we immediately arrive to a new momentum

$$p = m_0 \frac{v}{\sqrt{1-(\frac{v}{C})^2}} + qA$$

and after the usual transformations we obtain the Hamiltonian

$$H = C\sqrt{(m_0 C)^2 + (p - qA)^2} + qV.$$

Next, we can replace $p = -i\hbar\nabla$ and propose the quantum equation

$$i\hbar\partial_t \psi = H\psi$$

This last step goes over a lacuna, it follows a recipe that emerged in other works and has been adopted as a mechanised procedure. The presentation rests upon the hypothesis of a pointlike particle that is not even mentioned as a conjecture, following the procedure used by de Broglie, Pauli, Dirac, Einstein (but not by Lorentz, Abraham and others). There are theorems that tell us how to produce a unique square root of a positive operator, there are no doubts about the (positive) sign of the square root either. There are precedents of this sort of approach, for example in Pauli's attempts [Pauli Jr, 1927] to produce a quantum mechanics (in his case non relativistic).

If we insist with the new formulation and rewrite it as

$$\left[(-i\hbar\partial_t + qV) + C\sqrt{(m_0 C)^2 + (p - qA)^2}\right]\psi = 0$$

we observe that its solutions are also solutions with positive energy of

$$\left[-\left(i\hbar\partial_t - qV\right)^2 + C^2\left((m_0 C)^2 + (p - qA)^2\right)\right]\psi = 0$$

which is Klein-Gordon's equation *with the restriction to positive energies* (a similar proposal can be found in [Ch. XX, eq.XX.30 Messiah, 2014]). True, this equation does not predict positrons, but the predictions in physics given by equations subject to free interpretation have to be suspected to be affected by a known psychological effect: the frequency illusion or *Baader–Meinhof phenomenon* [Giacometti, 2022] and related illusions such as confirmation bias. Such illusions are the price paid for avoiding the rational discipline of the *epojé* (see Footnote 29).

We can produce the equations but the cost is to introduce arbitrariness (irrationality) by selecting inconsistent forms. Success is sustained by a biased narrative that conveniently forgets to tell us the failures of the approach and the prices paid. We do not subscribe these speculations, they are here to illustrate by contrast the strength of a rational approach, an approach that allows for growth and improvement.

5.D. Scholium. Are photons an instrumentalist trick?

Perhaps the reader would doubt our sanity if we tell them that bananas come in nature in mouthpiece chunks. At the end, they will find a banana tree, cut a banana bunch, separate one banana, peel it off and convince us we are wrong. However, this is not the case with magic bananas. Magic bananas are inaccessible to our senses except to taste while they are in our mouth. They have the same salutary benefits that natural bananas, yet we cannot find a tree of magic bananas and all what follows. The magic banana is that thing that we can taste, and recognise as banana, and we can digest. There exists no form of sensory access to the magic banana other than tasting one (or is it one chunk?) that is already in our mouth. Magic bananas are the prototype of photons, as you have certainly guessed. We know about photons only through their interaction with electromagnetic centres of action, called matter. Very much as the state of a quantum particle, it appears as we can make almost anything of it provided we are not measuring, i.e., not checking what we say. In both cases we are talking about what cannot be known, the perfect place for deploying our fantasy. The place has a name: unreality, a place where ideas cannot be related to the senses. In the unreal world magic bananas may exist.

The instrumentalist however will raise an important objection: the assumption of photons explains several experiments. The traditional experiments are the photoelectric effect, the Compton effect and more recently laser cooling. All of these experiments present some features in common: a high energy light beam is shined towards an electrically neutral piece of matter producing a mechanical effect. In the photoelectric effect it is an electron pulled from the material, in the Compton effect it is a change in the momentum of the target or a portion of it as well as scattered light and in laser cooling there is a reduction of the velocity relative to the laboratory.

The photoelectric effect (the only one of the three known by that time) represented a challenge for electromagnetism as it stood by 1905. The only known force in electromagnetism was the Lorentz' force and it could not explain the observations, something was missing in the theory. In 1905 Einstein perceived that the two problems haunting electromagnetism could be solved by the introduction of the photon. On one side it is a quasi particle and as such it allows to use imagination in the representation of action at distance (see 1.A.6), the only alternative to the ether available when thinking through analogies. If the photon carries energy, an heuristic explanation for the photoelectric effect was possible [Einstein, 1905b] and as we know we could get rid of the ether [Einstein, 1905a] at the small price of breaking the unity of science and changing the meaning of understanding (see Section 2.A). The photoelectric effect operates in another energy scale, about $10^{-2} - 10^{-3}$ of that in Compton scattering. The energy in the incoming beam exceeding the ionisation energy of the material contributes to the kinetic energy of the detached electron. No outgoing EM wave is recorded, so there is no actual scattering and in addition there is an electrostatic potential not present in Compton scattering. Otherwise, the energy exchange between field and matter corresponds to the one that will be presently described in Subsection 5.D.1. While the energy exchange with the microscopic system occurs in quanta, there is no need to force the EM field to be quantised.

Compton [1923] produced a theory to explain the scattering of X and gamma rays by light elements (see [Compton, 1921, 1922, Compton and Simon, 1925]), a phenomenon that could not be explained by an old theory put forward by Thomson [p. 325, 1903]. After a series of assumptions such as that the photon interacts with a single electron in the light element and other "free" assumptions, Compton manages to get a formula relating the frequency of the incoming and outgoing EM waves that matches experiments. In instrumentalist terms, we can indulge all the assumptions and manipulations made where we have no access to data and proclaim the result true. Modern experiments on laser cooling [Phillips, 1998] demonstrate beyond any doubt that some mechanical momentum is transferred from the light beams to the neutral atom in the interaction.

Undoubtedly, the idea of the photon is useful, but with the same certainty we can say that it breaks even further the unity of physics. The challenge is then to show that it is possible to explain the mentioned effects without introducing a quantification of the electromagnetic fields, or what is the same, that the Lorentz force is an incomplete form of the forces emerging in the interaction of light and matter.

5.D.1. Electromagnetism with quantum appearance. Our first task is to write Maxwell's equations in a form that resembles quantum theories, this is, using only positive frequencies, complex wave functions and incorporating Planck's constant. We begin by writing Maxwell's equations as an initial condition problem:

$$\begin{aligned} \frac{1}{C^2}\frac{\partial E}{\partial t} &= \nabla \times B' - \mu_0 j \\ \frac{\partial B'}{\partial t} &= -\nabla \times E \\ \epsilon_0 \nabla \cdot E &= \rho \\ \nabla \cdot B' &= 0 \end{aligned}$$

To simplify the bookkeeping we scale $\tau = Ct$ and $B = CB'$, so that the first two equations are rewritten as

$$\begin{aligned} \frac{\partial E}{\partial \tau} &= \nabla \times B - (C\mu_0) j \\ \frac{\partial B}{\partial \tau} &= -\nabla \times E \end{aligned} \tag{5.D.1}$$

We then set

$$\phi(x,\tau) = (E + iB)(x,\tau)$$

which solves

$$\begin{aligned} i\frac{\partial \phi}{\partial \tau} &= \nabla \times \phi - i(C\mu_0) j \\ \nabla \cdot \phi &= \frac{\rho}{\epsilon_0} \end{aligned} \tag{5.D.2}$$

Notice that the operator $\nabla\times$ is Hermitian and has eigenvectors of the form

$$\phi_k(x,\tau) = \exp(ik\cdot x)\left(\bar{E}_k + i\bar{B}_k\right)$$

where $\bar{E}_k, \bar{B}_k$ are complex functions of k,τ and

$$ik\times\left(\bar{E}_k + i\bar{B}_k\right) = \lambda_k\left(\bar{E}_k + i\bar{B}_k\right)$$

being $\lambda_k \in \mathbb{R}$ the frequency (eigenvalue) associated to the eigenvector. In matrix notation $ik\times = i\begin{pmatrix} 0 & k_2 & -k_3 \\ -k_2 & 0 & k_1 \\ k_3 & -k_1 & 0 \end{pmatrix}$ and $(ik\times)^\dagger = (ik\times)$. The homogeneous problem ($j=0,\ \rho=0$ in eq. (5.D.2)) is solved when

$$\begin{aligned} k\cdot\left(\bar{E}_k + i\bar{B}_k\right) &= 0 \\ k\times\bar{E}_k &= \lambda_k\bar{B}_k \\ -k\times\bar{B}_k &= \lambda_k\bar{E}_k \end{aligned}$$

which leads to

$$k\times\left(k\times\bar{E}_k\right) = -\lambda_k^2\bar{E}_k = -k^2\bar{E}_k$$

and the eigenvectors can be written as: $\bar{X}_k = \left(\bar{E}_k + i\hat{k}\times\bar{E}_k\right)/2$ with eigenvalue $|k|$ and $\bar{Y}_k = \left(\bar{E}_k - i\hat{k}\times\bar{E}_k\right)/2$ with eigenvalue $-|k|$. In other words, the matrix $\hat{X}_k = \left(Id + i\hat{k}\times\right)/2$ projects any complex vector onto the space associated with the positive eigenvalue $\lambda_k = |k|$, and $\hat{Y}_k = \hat{X}_k^*$ projects onto the space associated to $\lambda_k = -|k|$ while $\hat{X}_k^\dagger = \hat{X}_k$ and $\hat{Y}_k^\dagger = \hat{Y}_k$. In what follows the problem is restricted to solutions with $\nabla\cdot E = 0$ (transversal fields with $k\cdot\bar{E}_k = 0$)[73].

In conclusion, eq. (5.D.2) can be Fourier transformed in space, next the positive frequencies selected by multiplying by $\hat{X}_k$ and the resulting equation solved as the sum of the general solution for the homogeneous problem and a solution for the inhomogeneous problem. We can set the inhomogeneous solution to zero since it is not associated to the source of the light shined onto the atoms or emitted by them. The solution of eq. (5.D.2) in $k-$space reads then

$$\bar{X}_k(\tau) = \int_{-\infty}^{\tau} ds\left(-C\mu_0 j_\perp^+(k,s)\exp\left(-i|k|\left(\tau - s\right)\right)\right)$$

where

$$j_\perp^+(k,\tau) = \frac{1}{(2\pi)^3}\hat{X}_k\int d^3x\left[j\left(x,t\right)\exp\left(-ik\cdot x\right)\right] = \hat{X}_k\left[j_\perp\left(k,\tau\right)\right]$$

with $j\left(x,t\right)$ a neutral current ($\nabla\cdot j = 0$). Notice that since $j_\perp\left(k,\tau\right)^* = j_\perp(-k,\tau)$, being a Fourier transform of a real vector, we have that

$$\left[j_\perp^+\left(k,\tau\right)\right]^* = \hat{X}_k^*\left[j_\perp\left(k,\tau\right)^*\right] \quad = \quad \hat{Y}_k\left[j_\perp\left(k,\tau\right)^*\right] = \hat{Y}_k\left[j_\perp\left(-k,\tau\right)\right] = j_\perp^-\left(-k,\tau\right)$$

[73]The complete solutions of the homogeneous eq. (5.D.2) are the well known electromagnetic waves, $\phi_k(x,\tau) = \left(\bar{E}_k + i\bar{B}_k\right)_0\exp\left(ik\cdot x - i\lambda_k\tau\right)$.

a relation that will be useful later. If the light shined onto the atom has a well defined frequency, the time integral in essence selects $|k|$ values that match the frequency of the beam. Hence,

$$\bar{X}(k,\tau) \simeq -C\mu_0 j_\perp^+(k,0)\exp\left(-i|k|\tau\right)$$

is the incoming field.

We can then consider the energy of the field $\epsilon_0|\bar{X}(k,\tau)|^2 = \hbar\nu_k N_k$, with $N_k = \epsilon_0 \dfrac{|\bar{X}(k,\tau)|^2}{\hbar\nu_k}$ nonnegative real numbers describing the characteristics of the experimental setup. The electromagnetic energy can be written as

$$\begin{aligned}
\mathcal{E}_{EM} &= \frac{1}{2}\epsilon_0\int d^3x\phi^*\phi \\
&= \frac{1}{2}\epsilon_0\int d^3k\left[\bar{Y}^*\cdot\bar{Y}+\bar{X}^*\cdot\bar{X}\right](k,\tau) \\
&= \int d^3k\,\hbar\nu_k\left[\frac{1}{2}\epsilon_0\frac{\bar{Y}^*\cdot\bar{Y}+\bar{X}^*\cdot\bar{X}}{\hbar\nu_k}\right](k,\tau)
\end{aligned}$$

where the square bracket is another form for N_k as a consequence of the identity

$$\begin{aligned}
\int d^3k\,|\hat{X}_k\left(j(k,\tau)\right)|^2 &= \int d^3k\left[|j_\perp^+(k,\tau)|^2\right] \\
&= \int d^3k\left[|j_\perp^-(k,\tau)|^2\right] = \int d^3k\,|\hat{Y}_k\left(j(k,\tau)\right)|^2.
\end{aligned}$$

There is no reason whatsoever to force the N_k's to be integers. At this point, as a consequence of the experimental results we associate a momentum to the incoming field in the form

$$p = \int d\Omega(k)N_k\hbar k$$

or, for a more general case (generalising in Maxwell's style)

$$p = \int d^3kN_k\hbar k$$

In turn the identity $\int d^3k\,|\hat{X}_k\left(j(k,\tau)\right)|^2 k = -\int d^3k\,|\hat{Y}_k\left(j(k,\tau)\right)|^2 k$ gives

$$p = \frac{1}{2}\epsilon_0\int d^3k\ (\hbar k)\ \frac{\left[-\bar{Y}^*\cdot\bar{Y}+\bar{X}^*\cdot\bar{X}\right](k,\tau)}{\hbar\nu_k}$$

Thus, in the asymptotic regime so far considered when the radiation is not acting upon the atom we have $p_{,t} = 0$. However, Compton's theory describes the "scattering of light rays by light elements" and in general the wavelength and ray direction of the incoming light are different from those of the outgoing light.

Let us explore the electromagnetic meaning of p further. Recalling that $\hat{Y}_k = \hat{X}_k^*$ and thus $\bar{Y} = \bar{X}^*$ and $\left(\bar{X}+\bar{Y}\right)^* = \left(\bar{Y}+\bar{X}\right)$ we have

$$p = \frac{1}{2}\epsilon_0\int d^3k\ (\hbar k)\ \frac{\left[\left(\bar{X}+\bar{Y}\right)\left(\bar{X}-\bar{Y}\right)\right](k,\tau)}{\hbar\nu_k}$$

The expression $\frac{(\bar{X}-\bar{Y})}{-i\nu_k} = \int_{-\infty}^{\tau} ds\, \bar{E} = -\bar{A}$ (assuming $A(x,-\infty)=0$ and recalling that $\tau = Ct$), while $\left(\bar{X}+\bar{Y}\right) = \bar{E} = -\bar{A}_{,t}$. Hence

$$r \cdot p = -\frac{1}{2}\int d^3x \left(A \cdot (r\cdot\nabla) E\right)$$

for any arbitrary constant (in space) vector r.

5.D.2. Matter-field interaction. Having identified the mathematical object within electromagnetism that is involved in the standard approach to Compton's effect we proceed to look at the problem from the point of view developed in this book.

There are two electromagnetic fields associated to the same material object. The beam shined on matter produces a current in matter, the current is the consequence of the field. Hence causation is represented by the advanced solutions (see Subsection 4.3.1); the beam can be perceived and it is generated before the current. The vector potential satisfies the equation

$$\Box A_1 = -\mu_0 j$$

The induced current then is responsible for new radiation represented by a vector potential which now is a retarded integral of

$$\Box A_2 = -\mu_0 j$$

We can then say:

LEMMA 5.6. *In the case of scattering radiation there are two vector potentials, the one associated to the incoming radiation represented by advanced solutions of* $\Box A = -\mu_0 j$ *(say* A_1*) and the outgoing (scattered) radiation that solves the same equation but with retarded potentials (say* A_2*). We assume further that* $\nabla\cdot A_1 = \nabla\cdot A_2 = 0 = \nabla\cdot j$*, i.e., we work in the Maxwell-Coulomb gauge. In such a situation we have*

$$0 = \int d^3x \left[\epsilon_0 \left(A\cdot(r\cdot\nabla)E\right)_{,t} - 2r\cdot(j\times B)\right]$$

for any arbitrary constant vector r*, being* $A = A_1 + A_2$*,* $B = \nabla\times A$ *and* $E = -\partial_t A$*.*

PROOF. The proof is straightforward.

$$\begin{aligned}
\delta\mathcal{L} &= \int d^3x \left[-\epsilon_0 E\cdot((r\cdot\nabla)E) + \frac{1}{\mu_0} B\cdot((r\cdot\nabla)B)\right] = 0\\
\delta\mathcal{L} &= \int d^3x \left[\epsilon_0 A_{,t}\cdot((r\cdot\nabla)E) + \frac{1}{\mu_0} A\cdot(r\cdot\nabla)(\nabla\times B)\right]\\
&= \int d^3x \left[\epsilon_0 A_{,t}\cdot((r\cdot\nabla)E) + \frac{1}{\mu_0} A\cdot(r\cdot\nabla)(\mu_0 2j + \mu_0\epsilon_0 E_{,t})\right]\\
&= \int d^3x \left[\epsilon_0\left(A\cdot(r\cdot\nabla)E\right)_{,t} - 2r\cdot(j\times B)\right]
\end{aligned}$$

□

COROLLARY 5.3. *The force exerted on the material in the direction r is then*

$$r \cdot \int d^3x\ [(j \times B)] = \frac{1}{2} \int d^3x \left[\epsilon_0 \left(A \cdot (r \cdot \nabla) E\right)_{,t}\right]$$

and the total momentum is constant.

PROOF. The first statement is just the Lemma. Being p the momentum of the wave, $F = \int d^3x\ [(j \times B)]$ and p_e the momentum of the material (say an electron), we have

$$\begin{aligned} F &= (p_e)_{,t} = -p_{,t} \\ (p_e + p)_{,t} &= 0 \end{aligned}$$

□

In the standard discussion of the phenomenon, the latter result is assumed by analogy with mechanics and its relation with Lorentz' force is completely absent. The present result substitutes the ad-hoc hypotheses made by Compton and since then reproduced by textbooks. It links the formula to the theory showing that there is no contradiction between them.

In addition to Corollary 5.3 we have the conservation of energy presented in Table 5.2.2, namely $(\mathcal{E}_k + \mathcal{E}_{EM})_{,t} = 0$, which substitutes another ad-hoc hypothesis. From Table 5.2.2 we find that

$$\begin{aligned} (p_e)_{,t} &= \langle q\rho E + j_e \times B \rangle \\ (\mathcal{E}_k)_{,t} &= \langle j \cdot E \rangle \end{aligned}$$

and further $p_e = \langle -i\hbar\psi^* \nabla \psi \rangle$ and $\mathcal{E}_k = \langle -\frac{\hbar^2}{2m} \psi^* \Delta \psi \rangle$. The relationship between momentum and kinetic energy in the context of the experiment is an open issue.

In Table 5.D.1 we compare the hypotheses introduced by Compton to describe the phenomenon with those in the present discussion. In the conditions of the Table, both the present formulation and Compton's description coincide in the reading of the experiment. However, two of Compton's hypotheses are actually theorems in this presentation (marked with "No(*)"), while two other are not currently deduced from basic EM or QM principles (marked "Open"). For low target velocities in the EM limit, the present approach reproduces Compton's results without requiring quantisation of the electromagnetic field. Adopting in general the two hypotheses of Compton for the open issues, we also recover the conventional results. However, they are hypotheses: For a more demanding epistemology than the instrumentalist one, the issues are still regarded as open.

In contrast with the standard presentation, there is no need to consider photons, much less any kind of quantisation of electromagnetic fields. The photon is a fantasy, it has the same entity as the ether. In terms of producing formulae for the Compton effect (and other related observations such as the photoelectric effect) it operates as a didactic transposition that justifies the use of equations that originally belong in another area of physics –mechanics– where students have been habituated to them.

Hypothesis	Compton	Present
Wave carries momentum	Yes	Yes
Quantified field (photons)	Yes	No
Momentum conservation by analogy with CM	Yes	No (*)
Classical description of target	Yes	Open
Relation momentum–energy from SR	Yes	Open
Conservation of energy	Yes	No (*)

Table 5.D.1. Comparison of hypotheses. Compton: [Compton, 1923]. Yes/No indicates whether the ad-hoc hypothesis was made or not. (*): Not ad-hoc hypotheses but valid (true) results from the general formulation. Open indicates that it needs further investigation and temporarily Compton's hypothesis can be adopted. CM: Classical Mechanics.

5.E. Scholium. Mutilating reason

Let us repeat first some of Peirce's lessons. Logic (and mathematics) are subsidiaries of reason. Reason is self improving. (True) beliefs conduct our behaviour. Beliefs have to be dropped in front of empirical evidence contrary to them. When a belief falls, doubt sets in and begins a new theorising cycle.

When we admit "free invention", this is, invention not grounded on experience and current beliefs, we cut the flow of error backwards through the logical and mathematical structure and prevent the doubt to reach our beliefs. We become dogmatic. What was accepted as true cannot be changed. In this form we achieve a form of progress that never retraces its steps, needles to say, it is a sort of religious progress that makes us increasingly dogmatic. Dogmatism is the enemy of critical thinking. Critical thinking precisely seeks to find in the precedent the causes of the errors or surprises of the present. Once some wrong idea is found, the rational person feels compelled to revise all other ideas where the incorrect belief was used. If we find the ether does not exist, we have to revise all the results that have been obtained using the assumption of its existence. For some of us, this is a mandate of intellectual honesty.

However, if we derogate Truth and enthrone Utility, then we derogate intellectual ethics as well. As Kragh describes Dirac's views in his young years: "...the value of quantum mechanics lay solely in supplying a consistent mathematical scheme that would allow physicists to calculate measurable quantities. This, he claimed, is what physics is about; apart from this, the discipline has no meaning" [Kragh, 1990, p.80] (cf. Section 3.4). There is nothing else than utility, no Truth, much less ethics, after all, we all know "business is business", which means that business shall not be considered/judged from any other system of categories.

Learning from errors and improving ourselves and mankind by so doing is one of the most basic human mandates. Without it humanity would had never progressed, even more, surviving requires learning from errors, so animals do it.

The mutilation of reason, making impossible its critical motion, and leaving alive only the instrumental use of reason [Horkheimer, 1947] make us irrationals, inferior to animals in the rank of reasoning, actually, at the level of machines, since logical machines can compute and can collate data, yet they cannot (and actually, we cannot) tell how they do it, they cannot asses the quality and the forms of their "thinking". Reasoning is controlled thinking and it is this quality control what is dropped in "free thinking", actually utilitarianism.

When several generations have been indoctrinated in instrumentalism, the possibility of recovering reason is dim. It could only happen at the margins of the scientific society or a as consequence of catastrophic events. A convergence of dehumanised humans with logical machines in contrast seems completely plausible.

CHAPTER 6

The need for reason and a few rules

A word about this chapter

In the process of developing the present research we had to keep different levels of surveillance over our thinking. We were persuaded from the beginning that the process of developing our knowing was only possible by the praxis of knowing coupled to a higher process that took our own practice for observation and eventually for knowing about our knowing. The process was enriched by the reading of authors that somehow we felt related to our form of knowing. Among them, Charles Peirce, a completely unknown author for us when the research began, became one of our favourites. It was our friend Alejandro Romero who perceived the possible affinity and brought up his name as well as other authors such as Husserl. In the same form, at the beginning we did not know about the No Arbitrariness Principle or the letter exchange between Clarke and Leibniz. The principle appeared first and only later the precedents. This pattern repeated itself continuously during the research: what we thought was coming from inside had been entertained by authors that we had yet to read. Two suspicions emerge then: either ideas float in the civilisation and are transmitted in subtle ways we fail to recognise or there is a basic mechanism that we share as humans (or better: as animals) by which, in front of similar intellectual stimuli we produce like ideas. The first hypothesis explains nothing, the second has been repeatedly considered and given a name: reason.

The question was then: besides the rejection of arbitrariness, which other rules are built in us? Which are the rules of reason? Which are the early warnings that put us on alert in front of an ill constructed thought or a lie? This chapter tells the results of this exploration, we write them not as a "definite guide for rational thinking" but rather as a modest contribution to a research that humanity has postponed in excess. In this presentation we will use as examples matters that we have discussed previously by their own interest although they are now approached from a different perspective. We hope the reader do not find it reiterative.

6.1. Introduction

Abduction or retroduction[74] is indissolubly linked to the name of Charles Peirce (1839-1914) who studied scientific thinking from his pragmaticist perspective. Among his influential readings, Peirce indicates Aristotle [CP 1.22] [Peirce, 1994, CP 1.22 and several other paragraphs], Kant [CP 1.4] and Hegel, about whom he said “My philosophy resuscitates Hegel, though in a strange costume.” [CP 1.42]. Peirce’s works must then be contextualised within the Enlightenment. In [CP 2.191][75], he writes

> In studying logic, you hope to correct your present ideas of what reasoning is good, what bad. This, of course, must be done by reasoning; and you cannot imagine that it is to be done by your accepting reasonings of mine which do not seem to you to be rational. It must, therefore, be done by means of the bad system of logic which you at present use.

This is a declaration that conforms with the main requisite of the Enlightenment according to Kant’s view of Enlightenment [Kant, 1783] and somehow reminds us of Wilhelm von Humboldt’s view [Humboldt, 1792 Printed 1854, p. 20]:

> Whatever man is inclined to, without the free exercise of his own choice, or whatever only implies instruction and guidance, does not enter into his very being, but still remains alien to his true nature, and is, indeed, effected by him, not so much with human agency, as with the mere exactness of mechanical routine.

Peirce’s ideas about the process of grasping what knowledge means is very much exemplified in his own work, which –as much as the work of Aristotle [CP 1.22]– is *evolutionary*. This has been described as successive changes in Peirce’s view of abduction (shifting occasionally the name from hypothesis making, to induction, abduction and retroduction; see e.g., [Paavola, 2005]).

Since in Peirce thought is promoted by doubt and comes into a (temporary) rest when belief is reached, we must conclude that Peirce died leaving his work unfinished. Those, like the present authors, that attempt to (somehow) continue his work must be aware of the fact that in every continuation of Peirce’s thoughts there is something that comes from the reality of knowledge, something that comes from Peirce and something that is provided by the new subjects. There are at least two forms of continuation so far attempted, that have been called “Hansonian and Harmanian abduction” [Paavola, 2006].

Those authors that relate with Peirce by their inclinations as logicians emphasise the early syllogistic approach as in [Redding, 2003], where a correspondence is made among syllogisms in Aristotle, Hegel and Peirce. Authors inclined towards logic relate abduction to the following structure [CP 5.189]:

---

[74] Following Peirce, we use these words interchangeably. See however [Peirce, 1994, CP 1.65].

[75] To avoid extensive repetition, we will often quote Peirce by indicating the paragraph (e.g., [CP 1.65]) in [Peirce, 1994].

> The surprising fact, C, is observed; But if A were true, C would be a matter of course, Hence, there is reason to suspect that A is true.

The reasoning grants the proposal of A the name of *abductive inference*. However, to raise this expression to a definition leaves abduction abandoned to free interpretation. It must be recalled that for Peirce, [CP 2.195],

> Logic came about for the sake of reasonableness, not reasonableness for the sake of logic.

This is: the main concern is reason, not logic (as Paavola emphasises in [Paavola, 2004]). This view directs us towards Hansonian abduction and away from Harman's views.

The adoption of explanatory hypotheses is indissolubly linked to the adoption of beliefs and the cessation of doubt, the latter being the motor of thoughts. Peirce is clear in distinguishing the beliefs he pursues from religious and other common beliefs; he offers three characteristics of them:

> First, it is something that we are aware of; second, it appeases the irritation of doubt; and, third, it involves the establishment in our nature of a rule of action, or, say for short, a **habit**. As it appeases the irritation of doubt, which is the motive for thinking, thought relaxes, and comes to rest for a moment when belief is reached. But, since belief is a rule for action, the application of which involves further doubt and further thought, at the same time that it is a stopping-place, it is also a new starting-place for thought. [CP 5.397]

All these elements are essential to Peirce's notion of belief and as such, none of them can be dropped (not even in didactic examples) when regarding abduction, which is an act of thought. Further, Peirce sustains,

> [...] Thought ['s] sole motive, idea, and function is to produce belief, and whatever does not concern that purpose belongs to some other system of relations. [CP 5.396]

Thought, doubt and belief refer to our inner senses. They are to some degree defined in relation to each other. Abduction or retroduction in Peirce is also a part of scientific thinking. He provides more insight into it:

> These three kinds of reasoning are Abduction, Induction, and Deduction. Deduction is the only necessary reasoning. It is the reasoning of mathematics. [...] Induction is the experimental testing of a theory. The justification of it is that, although the conclusion at any stage of the investigation may be more or less erroneous, yet the further application of the same method must correct the error. The only thing that induction accomplishes is to determine the value of a quantity. It sets out with a theory and it measures the degree of concordance of that theory with fact. It never can originate any idea whatever. No more can deduction. **All the ideas of**

> **science come to it by the way of Abduction. Abduction consists in studying facts and devising a theory to explain them. Its only justification is that if we are ever to understand things at all, it must be in that way.** [CP 5.145] [emphasis added]

Further, the abduction of hypotheses in Peirce has as sole goal to put the hypotheses to test, and consequently to abandon them the moment they are refuted, continuing with the search for new hypotheses. This reflects in a cyclic or iterative process [Peirce, 1994, CP 7.220] where the subsequent proposal of hypotheses occurs from a different ignorance level: At least we know that all the previous attempts were inappropriate and we know where they first failed in the testing phase. In other words, the abductive reasoning does not "stop" at the first hunch but rather when belief is attained and a new and richer theory is proposed.

Dualism, reason and retroduction. Kant's dualistic view about the emergence of knowledge [Kant, 1787] through the coparticipation of understanding and intuition was further elaborated by W. Whewell [Whewell, 1858] as the "fundamental dialectic"

> [I]n all human KNOWLEDGE both Thoughts and Things are concerned. In every part of my knowledge there must be some *thing* about which I know, and an internal act of *me* who know. [...] [Man is] *interpreting* the phenomena which he sees. He often interprets without being aware that he does so.

and further propagates in time into the views of Piaget and García [Piaget and García, 1982]

> Un hecho es, siempre, el producto de la composición entre una parte provista por los objetos y otra construida por el sujeto [Original version].
>
> A fact is always the product of the composition between one part provided by the objects and another constructed by the subject [Our translation].

There is no evidence of Peirce being deeply aware of the intervention of the subject in the production of facts. Peirce permanently refers to "observed facts", hence ignoring the participation of the subject in the production of facts alongside what comes from the external senses. We found only an indication of awareness in paragraphs concerning self-consciousness as a result of a clash between an outward and inward motion [CP 8.41]. This absence represents a substantial drawback that needs to be repaired since it is in this process where –according to Piaget and García as well as Husserl [Husserl, 1983]– the process of ideation occurs, this is, where the observed is registered as a fact after (quite often unconscious) rationalisation. The introduction of a dualist vision of knowledge (Chapter 2) makes enough room to incorporate pre-rational elements such as (simple) intuition, habits, epistemological frames, preexisting theories, phantasy, imagination and in general the elements that participate in the initial production of facts but belong to the subject and not to the object[76].

These pre-rational elements participate along with reason in the process, yet, in the ideal final product where the phases of abduction, deduction and induction as described by Peirce can be identified, the contributions of the "turbid" thoughts have been made transparent by reason, leaving only the distinctive clear thinking. There will be room as well for the flux of error and the iterative refinement of theories.

Little can be gained however if we cannot state objectively what is meant by reason. *Scientific*, for Peirce, means here what really directs us to satisfy the aims of those "to whom nothing seems great but reason" and see nature as "a cosmos, so admirable, that to penetrate to its ways seems to them the only thing that makes life worth living" [CP 1.43]. Thus, reason is essential to science and appears as associated to the making a cosmos from the chaos that reaches our senses.

We highlight that reason is a mandatory part of understanding, and that understanding is a name for having a theory in which we can believe, recalling that belief in Peirce establishes rules of action. We will advance in the present project producing rules for the control of the rationality in retroduction. For Peirce [CP 5.384] the concept of reality (see Section 6.4.1.1) is his sole hypothesis, although he claims that "...the method must be such that the ultimate conclusion of every man shall be the same". Putting things together, the current logic of any human should be of such kind as to allow them to improve it and reach an intersubjective level of understanding in all matters concerning reality. Peirce closes his paragraph with

> Experience of the [scientific] method has not led us to doubt it, but, on the contrary, scientific investigation has had the most wonderful triumphs in the way of settling opinion. These afford the explanation of my not doubting the method or the hypothesis which it supposes; and not having any doubt, nor believing that anybody else whom I could influence has, it would be the merest babble for me to say more about it. If there be anybody with a living doubt upon the subject, let him consider it.

Peirce's optimistic view is based upon his conception of the scientist. However, much of what is labelled as scientific today does not fully meet Peirce's criteria exposed in "The scientific attitude" [CP 1.43–1.45] but rather what is described in [CP 1.45] as non scientific. Thus, reason is essential to science, and what reason means in the present context needs to be further explained if the confusion that enters our minds through interests in conflict is to be avoided[77] .

We will adopt a constructivist approach influenced by Piaget's work. A reason without consciousness of its own constructive efforts will consider its scientific

[76] In this sense, [Paavola, 2005] recognises "a clearer change in Peirce's views than from evidential to methodological perspective concerned the role of instinct in abduction."

[77] We mean the utilitarianism indicated by Peirce in [CP 1.45] as well as the careerism imposed onto academics by governing/administrative bodies [Solari et al., 2016] and the social pressure exerted by the scientific field [Bourdieu, 1999].

activity to be the discovering of the laws of the universe, perhaps without noticing that their efforts are preformatted by their own rules of reasoning and their own criteria of considering an argument to be correct/convincing/acceptable. Thus, there is a level of metascientific criteria that needs to be explored and explained. We owe the idea regarding the existence of rules or norms to Piaget, as Gruber and Vonèche [1995, p. 739] write:

> Rules or norms are generally considered as dependent on structures in the subject. They do not depend on the structure of physical reality for their validation but are instead entirely determined by a principle of deduction that is not empirical in nature.

6.2. On Hegel, Peirce and rationality

The mysterious phrase (quoted in the introduction) regarding Hegel's philosophy in relation to Peirce's own philosophy [CP 1.42] deserves some inquire. We read in [Hegel, 2001a]:

> (§ 101) ... progress in philosophy is rather a retrogression and a grounding or establishing by means of which we first obtain the result that what we began with is not something merely arbitrarily assumed but is in fact the truth, and also the primary truth.
>
> (§ 102) It must be admitted that it is an important consideration —one which will be found in more detail in the logic itself— that the advance is a retreat into the ground, to what is primary and true, on which depends and, in fact, from which originates, that with which the beginning is made.
>
> (§ 1707) So far, then, it must be said that cognition, once it has begun, always proceeds from the known to the unknown.

These ideas correlate with [CP 5.189] (quoted in Section 6.1) as follows: what "we began with" is C. Thus, progress means to find the A that makes C true and not arbitrarily assumed and that is also grounded to our pre-existing knowledge base. Further, notice the occurrence of "retrogression", the backward motion implied as well in retroduction. Later, Hegel gives precision when discussing analytic and synthetic cognition:

> (§ 1720)... Analytic cognition is the first premise of the whole syllogism —the immediate relation of the Notion to the object; identity, therefore, is the determination which it recognises as its own, and analytic cognition is merely the apprehension of what is. Synthetic cognition aims at the comprehension of what is, that is, at grasping the multiplicity of determinations in their unity. It is therefore the second premise of the syllogism in which the diverse as such is related.

Peirce [CP 4.85] in turn will take (as much as Hegel) synthetic cognition from Kant [1787, p. 37][78]. Along the development of his ideas, Peirce [CP 2.629] writes :

> So, an hypothesis is really a subsumption of a case under a class.

If we consider that the analytic judgment carries no other novelty than making explicit what was already contained in the premises, the progress of science is linked to the synthetic judgment. And, if in addition, abduction is the "only logical operation which introduces any new idea" [CP 5.171], making the intersection of the claims, rational abduction is of the order of "ampliative abduction" (see for example [Aliseda, 2004]).

It is apparent that from Hegel to Peirce the grounding of the hypothesis has been lost or at least it has been de-emphasised. Explanation is not the only goal in Hegel as he wants to remove arbitrariness. An hypothesis which is arbitrary but explains the observed and is not refuted by other implied facts would not achieve the desired elimination of arbitrariness but only a translation of arbitrariness from the fact to the hypothesis. Since arbitrariness is "the quality of being based on chance rather than being planned or based on reason" (Cambridge dictionary) or "existing or coming about seemingly at random or by chance or as a capricious and unreasonable act of will" (Webster dictionary), it appears as a safe measure to request the abductive hypothesis to be rational. A requirement that only puts us at the beginning of a quest: how to determine intersubjectively the reasonability of an hypothesis?

If one is set to understand the "logic of science", which is certainly Peirce's intention, one is obliged to adopt abduction as implying not only the logical rule of [CP 5.189] but the additional requirements for hypotheses to be ampliative and rational. Peirce [CP 7.220] requests of the hypothesis first to "be capable of being subjected to experimental testing" and second it "must be such that it will explain the surprising facts we have before us which it is the whole motive of our inquiry to rationalize". Thus, our observation to Peirce is that rationalisation is not completely achieved by producing an explicative hypothesis, since such an hypothesis may very well be irrational itself.

Hanson [Hanson, 1965] distinguishes three ingredients in the logic of discovery

> 1.) proceeds retroductively, from an anomaly to
> 2.) the delineation of a kind of explanatory H which
> 3.) fits into an organized pattern of concepts

Thus, Hanson's view recovers the "grounding" present in Hegel but not so evident in Peirce. We will try in the coming sections to move towards the recovery of a self-critical rationality which is the ultimate exercise that allows us to improve our science. In section 6.4 we propose some minimalistic principles of rationality.

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[78] Notice that because of a difference in translations Peirce adopted "ampliative judgments" while the referenced paragraph says "augmentative judgments".

6.3. On the relation of abstraction and ampliative hypotheses

The word abstraction (from Latin *abstrahere* "to drag away, detach, pull away, divert;") can be read in at least two forms. A first form, procedural, in which we eliminate properties leaving a less determined, more general, idea. This procedure leads to the question: what is then left when all properties are withdrawn? Such thing has been named the "thing-in-itself", a metaphysical entity already criticised by Sartre[79] as devoid of any meaning. The problems with this procedure have been already indicated by Hegel [2001a, § 22]. A different connection with the etymology is to extract, to pull out its essence from the concrete form. Consider the first form, stopping before hitting emptiness. Let us say we start with an object and stop with a general idea of the object, an *abstraction* of it. Because of its sub-determination, chances are that other objects, after a similar operation match our abstraction as well. In this way, a relation is established between them: they belong to the same class (the relation is easily shown to be transitive). For example, suppose I am holding a purring thing in my arms and say: this is my cat. The same saying can be used by every cat owner despite all the possible differences among cats. We look now to what we have achieved: to put in relation the singular with the general, the particular with the universal. A particular cat is a cat because it can be put in relation with the abstract (general/universal) idea of cats while the general idea is such because it is something that can be put in relation with all the particular forms of cats. The idea of a cat cannot pre-exist the particular cats from which it is abstracted, but the particular cats are in no form recognised as cats if there is no general idea of them. Thus, in a cognitive sense, the abstract form and the particular realisations are created at the same time. Yet, the unrecognised things exist even if we have not yet recognised them as cats. This is an example of dialectical opening in which two or more concepts are produced simultaneously because such relation is useful to organise the observable input. In the particular case of abstraction, in its more general form corresponds to the dialectic "universal-particular" or the relation between the one and the multiple.

Incidentally, the Meno paradox has been considered in relation to abduction in [Paavola and Hakkarainen, 2005]. The paradox is:

> If you know what you're looking for, inquiry is unnecessary.
> If you don't know what you're looking for, inquiry is impossible.

[79]"Force, for example, is not a metaphysical conatus of an unknown kind which hides behind its effects (accelerations, deviations, etc.); it is the totality of these effects. Similarly an electric current does not have a secret reverse side; it is nothing but the totality of the physical-chemical actions which manifest it (electrolysis, the incandescence of a carbon filament, the displacement of the needle of a galvanometer, etc.). No one of these actions alone is sufficient to reveal it. But no action indicates anything which is behind itself; it indicates only itself and the total series. The obvious conclusion is that the dualism of being and appearance is no longer entitled to any legal status within philosophy. The appearance refers to the total series of appearances and not to a hidden reality which would drain to itself all the being of the existent. And the appearance for its part is not an inconsistent manifestation of this being. To the extent that men had believed in noumenal realities, they have presented appearance as a pure negative. It was "that which is not being"; it had no other being than that of illusion and error." [Sartre, 1966, Introduction]

> Therefore, inquiry is either unnecessary or impossible.
> Meno paradox (consulted December 14th 2022)

The dialectical opening means that we create the form (the idea) because in a pragmaticist sense it serves the purpose of organising the observable. The idea is born out of the inquiry itself. The duality does not pre-exist, it is created in a single unitary act and it is justified because of its appropriateness. Dialectical openings dissolve the possibility of applying the paradox since the paradox presupposes the existence of one term before the other.

Abstraction does not explain, it only organises, then abstraction is not abduction despite it being pragmatic as well. At the same time, theories cannot be made with respect to a unique event, the singular. Before deserving further study each experiment must be reproduced, and by reproduction it is understood not an impossible new production in exactly the same conditions (including time, space, personal, apparatus, ...) but rather the production of a new experiment related to the original by an abstract form. It is the abstract form what is reproduced[80].

Establishing this relation represents the essence of abstraction, the abstract form into which other forms of putting in relation can be mapped, as for example *cognitive surpass* [Piaget and García, 1982] (see Section 6.4).

Abstraction is not performed without guidance, when abstracting we have in consideration some matters of concern that lead our quest. For example, we all know that tigers are cats, but if what is in consideration is the ability of purring, then tigers are not that kind of cats as they do not have this ability because of physiological reasons. Thus, any particular being relates to a multitude of abstract forms.

The dialectic particular-universal pervades science. Every time we "discover" a "regularity" of nature we can say with identical precision that we have established a new abstract form for organising nature.

By the name *experience* we usually designate the construction of an abstract form that relates to a finite sequence of events. By the way of abstraction, events known in isolation become particular realisations of the same experience. Thus, experience is in part abstraction. Abstraction enacts the identification of the different. When the abstract form has been established we can ask whether new events correspond to this form or not. In case they do, we can translate to them, caring for their particularities, the results of theories that correspond to the abstract form.

Theories actually relate ideas, abstract forms, and it is this character of abstract what allows us to relate them to new particular events. An explanatory hypothesis introduced to explain only a singular event does not constitute a theory. When we propose hypotheses to explain a unique event but do not establish a belief that rules our future actions, we are not constructing theories, and we are

[80] It is not unusual for experiments to be irreproducible. In such cases what often happens is that the abstract experiment does not completely describe the determining circumstances. Some determining conditions have not been controlled, much less communicated, as they are outside what deserves control in terms of the theoretical background underlying the design and development of the experiment. Then, the theoretical background must be rejected, being considered at least incomplete.

not in the process where scientific retroduction belongs. The ground for advancing hypotheses is prepared by abstraction or must be performed alongside of, or before, abstraction.

But if abstraction is not abduction, and abstraction produces a new idea in our consciousness, how is it possible that abduction is the only creative moment of science? Since abduction and abstraction are synthetic judgments, the right to be called the only creative moment of science corresponds to the synthetic judgment. Have we come back to Kant?

Consider the equation of state of ideal gases,

$$PV = n(kT)$$

where P stands for pressure, V for volume, n number of moles, T is the absolute temperature and k is a constant such that (kT) represents energies. The equation has a clear abstraction since it applies to all diluted gases irrespectively of their chemical nature and the form of the container (at least). All diluted gases map into the same expression, but only in the limit of infinite dilution the expression matches experiments, hence it is an idealisation in the sense of Galileo [Galilei, 1914]. Finally, it is a synthetic relation. Each one of the elements introduces ideas into our consciousness and concurrently produce an interesting relation that explains observations. But idealisation is not a synthetic judgment.

Idealisation and abstraction produce indeed new ideas. However, they enter into explanation only through abduction. Thus, thinking in terms of the production of theories (explanations) the abductive step can be synthesised with abstraction and idealisation as they are complementary when the goal is understanding nature.

6.3.1. Abstraction and analogy.

> Analogy is the inference that a not very large collection of objects which agree in various respects may very likely agree in another respect. For instance, the earth and Mars agree in so many respects that it seems not unlikely they may agree in being inhabited. [CP 1.69]

Between two similar problems, analogy makes a direct connection. In contrast, abstraction makes indirect connections conditioned to the possibility of producing an universal form which can be particularised in each of the different problems originally perceived as potentially related. In so doing, abstraction opens the possibility of a manifold of connections other than those initially considered, largely enlarging the possibility of performing empirical contrastive comparisons. In the quoted example, if instead of producing the inference we propose a (tentative) abstract form consisting of the class of objects that have the identified set in common (call it *habitable planets*), the route just started leads us into attempting to prove that the characteristics selected to define the class can, by themselves, determine the possibility of life on the planet. If this is the case, the analogy is correct, if not, it is incorrect. The analogy lacks the element of rationality which is present in "clear thinking", this is, abstract thinking, thinking using abstractions.

6.3.2. Abstraction and phantasy. The English spelling phantasy (Greek: $\varphi\alpha\nu\tau\alpha\sigma\acute{\iota}\alpha$, Latin: *imaginatio*) is predominantly associated with "imagination, visionary notion" (Oxford). The Latin etymology of *imaginatio* relates it to *imitare* (to copy). According to Hume [2011, p. 7], we think in terms of elements pertaining to imagination which, according to Aristotle [Aristotle, 1907, p. 123, see 3.3–3.15], lies close to both perception (Impressions in Hume) and belief, but not to clear thinking. We further learn from Husserl [1983, Ch. One] that the production of ideas by intuition (eidetic seeing) can be triggered by both the real (observable) world as well as by phantasy. Very much like Aristotle, Husserl reminds us that whatever predicate having to do with "matters of fact" must be grounded on experience ("And thus not even the most insignificant matter-of-fact truth can be deduced from pure eidetic truths *alone*." Husserl [1983, p. 11]).

In short, concerning the study of nature (physics) there are two possible approaches at least since Aristotle: "clear thinking" and "phantasy". The former struggles for being correct while the latter may be correct or not, and its correctness cannot be established without the participation of clear thinking[81].

6.3.3. When abstraction was left behind. As the XIXth Century advanced, two forms of thinking had evolved without a neat distinction, differing in the relation among clear thinking and phantasy. Both of them using the word *idea* but meaning different things with it. The requisites for achieving belief, or the cessation of doubt are expected to be unlike as well. The subtleties of the different meanings may emerge when we push our reasoning to its limits. We call *imaginative thinking* to the form originating in the *Bild* concept [D'Agostino, 2004], where images (often called ideas) were the central tool in developing knowledge, and *abstract thinking* to the form supported by abstraction (cognitive surpass [Piaget and García, 1982, Introduction]).

Abstract thinking requires the development of abilities from within, in the form in which Peirce indicated that our current logic must approve its own improvement. Abstract thinking is then linked to W. von Humboldt's *bildung* (self formation) [Sorkin, 1983]. As it comes from within, self formation does not lend itself for massive education. In contrast, always according to Kant, the formation of professionals requires them to master the use of established ideas, to think with their master's ideas. The emphasis at the Prussian universities first, and later in the rest of Europe, changed from bildung into instruction [Helmholtz, 1908, VI On academic freedom in German universities]. The focus of higher education becomes not as much clear thinking but rather *certified belief*s.

The investigation of the influence brought about by the second industrial revolution into science deserves an independent inquire.

Poincaré [1913b, p. 185-186] summarised the situation at the beginning of the XXth Century:

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[81] Notice that the current use of fantasy (modern spelling) does not correspond well with Aristotle's use. Phantasy here does not mean fantasy as in day-dreaming for example but rather imagination as it is praised in [CP 1.46].

> Most theorists have a constant predilection for explanations borrowed from physics, mechanics, or dynamics. Some would be satisfied if they could account for all phenomena by the motion of molecules attracting one another according to certain laws. Others are more exact: they would suppress attractions acting at a distance; their molecules would follow rectilinear paths, from which they would only be deviated by impacts. Others again, such as Hertz, suppress the forces as well, but suppose their molecules subjected to geometrical connections analogous, for instance, to those of articulated systems; thus, they wish to reduce dynamics to a kind of kinematics. In a word, **they all wish to bend nature into a certain form, and unless they can do this they cannot be satisfied**. Is Nature flexible enough for this? [*Emphasis added*]

Concluding this section, it must be asserted that the desires of Peirce in [CP 5.384] for science to be a method upon which "the ultimate conclusion of every man should be the same" cannot be fully carried out in current scientific practice since the requirements for the cessation of doubt are possibly different for different persons. The fade out of reason [Horkheimer, 1947, Feyerabend, 1987] during the XXth Century has completely changed the scene.

6.3.4. Glossary. Before moving on we summarise our use of polysemous words:

- **Abstract** (verb): The mental activity that produces relations between the multiple particulars and the universal form. It creates the association and produces relations between the particulars.
- **Analogy**: The inference that objects which agree in various respects agree in another respects. It produces a direct relation between the analogous elements not mediated by an universal form.
- **Arbitrary**: Not produced by reason. In conflict with reason.
- **Clear thinking (or rational thinking)**: is a form of thinking in which each step is justified and controlled for arbitrariness and leaving neither hiatus, nor lacunae, nor turbid thinking (see below).
- **Critical thinking**: The reasoning directed towards the fundaments (as in [Hegel, 2001a, § 101]). The discovery of the hidden assumptions that frame our thoughts (epistemic frames, habits, simple intuitions, phantasies, ...) and their subsequent removal or rationalisation, making our thoughts clearer.
- **Phantasy/imagination**: An idea not firmly grounded in observations but necessary for our thinking. A posteriori they may be grounded, rejected or remain as phantasies. The negative sense usually associated to them corresponds to: something created by our mind to avoid questioning our theories or fundamental beliefs.
- **Reason**: The mental faculty which is used in adapting thought or action to some end; the guiding principle of the mind in the process of thinking.

Frequently contrasted with will, imagination, passion, etc. (Adapted from the Oxford dictionary).

- **Turbid thinking**. The construction of ideas by not completely conscious and rational methods such as: simple intuition, habit (usually unconscious), imagination/phantasy, analogy and the dogmatic use of epistemological frames and pre-existing theories.
- **Understand**: To have a rational theory that explains the matter in question.

6.4. Science, reality and the rules of rational retroduction

We have illustrated the existence of different approaches to the cessation of doubt, and consequently different *sciences*. They cannot be considered equivalent since one is based in clear thinking while the other resorts to non-rational ingredients. In order to develop a rational retroduction it is mandatory that the very notion of rationality becomes objective and intersubjective. We attempt here a proposal for the objective determination of rationality.

We further notice that for Peirce explanation can be equated to rationalisation as it is evident from the following quotation:

> I think I have now said enough to show that my theory – that that which makes the need, in science, of an explanation, or in general of any rationalization of any fact, is that without such rationalization the contrary of the fact would be anticipated, so that reason and experience would be at variance, contrary to the purpose of science – [that this theory] is correct, or as nearly so as we can make any theory of the matter at present. [CP 7.201]

6.4.1. Science and reality. The task of understanding involved in scientific theories requires some precision on what we mean by reason and the requisites for inference.

6.4.1.1. *The principle of reality.* In the first place, we must indicate that the attempt of constructing a cosmos out of sensorial input implies the assumption that there is something real that reaches us through the senses, this is to say, that there are subject and object. While the truth of this statement is debatable, we can consider the dangers involved in accepting or rejecting it. Little damage is done if accepting reality were an error and it turns to be that everything is part of a unique encompassing being. On the contrary, if we were in error when rejecting reality, we would become completely dysfunctional and miss one of the greatest opportunities in life.

This principle was addressed in Section 1.1 and in Section 2.3 as part of the discussion on Peirce's views. Reality in Peirce can be seen as a duality. In front of us we have what is perceivable (observable but not yet "observed") and what is elaborated by ourselves from this input: the ideated or ideal (*facts* in [Piaget and García, 1982]).

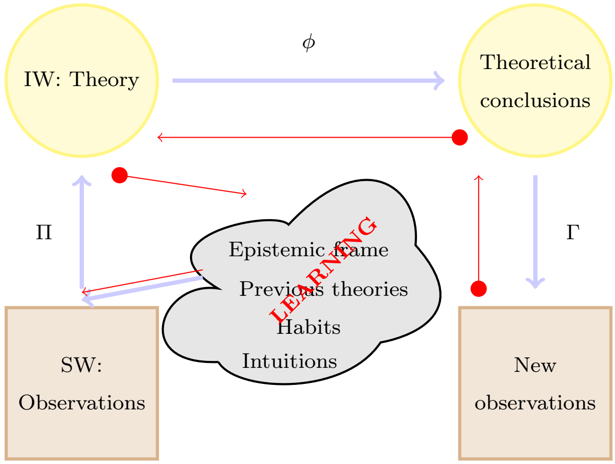

FIGURE 6.4.1. Schema for the progression of knowledge. Π is a projection that produces the real out of the observed, ϕ stands for a theoretical elaboration (which eventually can be empty, in such a case ϕ is the identity Id) and Γ is the interpretation that produces an expected observation. The red circuit indicates the effect of refutations, eventually leading to an improved version of Π .

The progression of knowledge. A schema for the progression of knowledge (elaborated from Figure 2.3.1) is depicted in Figure 6.4.1.

According to this schema, to grasp reality we need the Sensorial World (SW) that we perceive "out there", its ideated or intuited forms or ideal world (IW) which rest within us and a form of correspondence from one world with the other: the phenomenological map, Π, Γ. We also need to include in reality as a primary element the consciousness of our mental operations as they are not ideas produced by elaboration of input from the senses –they are neither in SW nor in IW– and are not part of the phenomenological map, but rather its producer[82]. In natural science we restrict the study to the development of the relation between IW and SW, something that is not possible if we want to study cognition or psychology ignoring self-consciousness. Notice that whatever is in IW, it is not what is sensed, it is clearly not in SW. Hence, we can think of IW as a negation of SW, and the dialogue through the phenomenological map between the two forms of the real constitutes what science is. We have included a cloud of "turbid thinking" in Figure 6.4.1. The turbid thinking provides suggestions on how to produce facts

[82]Thus, reason does not belong to the real world (SW, IW) but rather to an internal world which acquires other sort of reality in self-consciousness.

and theories from observations. Sooner or later these precursors of the final ideas must be brought to our conscience to be examined and depurated by reason. The situation becomes mandatory when the theory is refuted. The observable events cannot be changed; in contrast, the facts we associated to the observations must be changed. The rational option is not to doubt about reason but rather doubting the pre-rational elements. Out of the process more observations are sought and, hopefully a renewed (more rational) theory emerges. We usually say that we have learned from our mistakes. Finally, ϕ represent theoretical elaborations, basically ruled by mathematical logic.

6.4.1.2. *Scientific reasoning.* The casting of science into the forms of dialectics, performed in Section 6.3, is leaving out an important fact. The ideal world shares a fundamental characteristic with living things: it is self-reproducing, it entails creative production, *poiesis*. We call this reproductive act **reasoning**, an activity that produces new ideas out of previous ideas or observations. It is then the poiesis of our conscience what opens up the duality IW/SW. In other words, IW is a result of the dialectic corresponding to the opening Ego-SW, where Ego is the knowing and SW is the knowable. The destiny of IW is then to change permanently, ideas generating experiences and experiences generating ideas. Our beliefs are not stable, and every time we try to see whether they come from the reasoning ego or from SW we end up finding the opponent[83]. This ever changing IW is then part of the dialogue (*becoming* [Hegel, 2001a, § 134]) between Ego and SW, accounting in this form for the mixed character of experience and facts. Then, if reasoning is the activity that institutes new ideas in IW, the sensorimotor cognitive activity of children in their earliest times in full contact with SW must be considered as reasoning, since it institutes the idea of self (ego), and alter (not ego), the idea of the permanent (the identity, what remains unchanged through perceived changes) and the transition between states of permanency (change), at the same time they conceive space and time [Piaget, 1999][84]. These ideas frame all further knowledge. The development of early cognition just presented certainly belongs to IW and was developed by Piaget from hints obtained by the observation of SW. We can then say that it is reasoning what institutes the duality we call reality, and it does so through the sensorimotor activity of the child.[85]

[83]Reason casts doubts on observations and observations cast doubts on reasoning. They continually "negate" each other and strive to explain themselves to the other (to agree) to preserve our unity. In this "becoming" they construct reality. Beliefs are held when this becoming rests for a while.

[84]We return to the issue of pairs of opposite concepts in Subsection 6.4.2.2.

[85]Peirce [1994, CP 8.41] discusses Kant's a prioris of Space and Time concluding that something of the kind of Will, for which he finds no better word than **volition,** enters into these sensations. A sense of collision or clash between an inward and outward motion, a "consciousness of duality" or a "duality of consciousness". This duality leads to a criticism of Hegel: "The capital error of Hegel which permeates his whole system in every part of it is that he almost altogether ignores the Outward Clash". Peirce writes: "Feeling is simple consciousness" which can be linked to the "simple intuition" in [Husserl, 1983] and the consciousness of the duality to "philosophical intuition" (the "lower and higher consciousness" in Pierce). He recognises as well that "conceptions which are proved to be indispensable in Formal logic, must have been already rooted in the nature of the mind when reasoning first began". A thought in the same direction as Piaget's ideas and well aligned with Hegel [2001a, § 4] where "how to think" is put at the same autonomous level than digestion and moving. Thus, as an innate activity with no

In this sense, abduction is the kind of inference entering the construction of theories in science and not any kind of inference as discussed in Sections 6.2 and 6.3. According to the abstract mind, the cognitive activity we call science aims at the production of cognitive surpasses (see a detailed discussion in Section 6.4.2.3). Consequently we can call *scientific knowledge* to the outcome of this activity.

The notion of scientific understanding we have coined contrasts with the notions of assimilation and accommodation. Both these forms of cognition are present in the child's development. The idea of assimilation and accommodation belongs to the family of imitation [Gruber and Vonèche, 1995, Introduction] which is within the same realm as analogical thinking. It is present since early times in life and is in use by the time of the development of abstract thinking, likely a prerequisite and precursor. Abstract thinking is characterised by the cognitive surpass increasingly present in the adolescent formal thinking[86], since the universal is reached by form of abstraction which is to put the actual as a case of the possible, being then the possible the universal form of the actual.

6.4.2. On the relation between subjective, intersubjective and objective.

6.4.2.1. *The No arbitrariness principle.* If we introduce some arbitrary decisions in the scientific discourse (be it for the sake of the argument or with the aim of facilitating an explanation), the set of possible arbitrary elements must have the internal structure of a group[87], being then the set of all possible presentations of the argument a representation of the group and as such all of them equivalent. Further, it is shown that the facilitation of the relational concept of space due to Leibniz produced by the introduction of a privileged observer introduces a (useful) subjective element, the *subjective space* (the space in all elementary physics texts) along with a series of properties of this space as well as conditions that the statements regarding physical laws must satisfy if they are going to remain rational.

When constructing a theory we have to make an early decision: are we going to introduce arbitrariness or not? The decision has not much relevance if we keep track of the introduced arbitrariness, and acknowledge the necessity of (and the

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need for education in logic. All these philosophers appear to present particular expressions of an abstract idea not completely apprehended.

[86] "Formal thinking is both thinking about thought (propositional logic is a second-order operational system which operates on propositions whose truth, in turn, depend on class, relational, and numerical operations) and a reversal of relations between what is real and what is possible (the empirically given comes to be inserted as a particular sector of the total set of possible combinations). These are the two characteristics—which up to this point we have tried to describe in the abstract language appropriate to the analysis of reasoning— which are the source of the living responses, always so full of emotion, which the adolescent uses to build his ideals in adapting to society."[Gruber and Vonèche, 1995, p. 438]

[87] For example: We can say that the relations in the invariant relational space are lifted into relations in the subjective spaces by arbitrary decisions, but since the subjective statements must remain equivalent, there must be a group of transformations, T, that allows us to move from one presentation to the other. If we conceive now a theory as a space of statements, E, relating different concepts belonging to our subjective presentation, what is real in them is only the core that remains when we remove (mod out) the arbitrariness, $D = E/T$, which is the result of identifying statements that only differ by the introduced arbitrariness. Thus, D is invariant while E is equivariant with respect to T.[Solari and Natiello, 2018]

methods for) removing it. However, if we lose consciousness of our constructive effort, we might inadvertently enter into the realm of arbitrariness. No amount of mathematics will take us ever out of the subjectivist cage, since the necessary step is not an analytic/deductive judgment but rather a synthetic/critical one. This is, we need to understand not what the consequences of our beliefs are, but rather which is the foundation of our beliefs.

The rejection of arbitrariness is conceptualised in the No Arbitrariness Principle (see Section 1.1).

A criticism of empiricism. A property is a quality proper of something. Whenever there is a property, there is something to which it belongs. If s stands for something and p for property, the basic enunciation is: "s is p". The set of properties, $\mathcal{P}$ is the set of all possible values of p irrespective of the s. It is true that the enunciation "s is" (produced after elimination of all the properties) is meaningless, an argument that is found in [Carnap, 1959, The significance of a sentence], but it only indicates that the search for the essence by depriving the object of its attributes is the wrong path. In the same form in which we admit a set of properties, we are forced to admit the set of objects constituted by all those things, $\mathcal{S}$, pointed by s in statements of the type "s is p", regardless of the property p. Doing otherwise is an instance of arbitrariness, since –as already discussed– the universal of something is nothing but the set of all particular forms of the matter/object under consideration[88]. It is important to notice that here as well as in Sartre, the metaphysical (and Kantian) "thing-in-itself", the noumenon, is eliminated as in Mach and Carnap, but the abstract in Sartre survives.

The elements in $\mathcal{S}$ bear all of them in common an undeniable property which cannot be suppressed, a essential property in the words of Mach: all members of $\mathcal{S}$ can be used in statements of the form "s is p" for some well selected $p \in \mathcal{P}$. The set of well selected properties regarding an object s, call them $\mathcal{P}_s \subseteq \mathcal{P}$ is the bundle of properties associated to the object in Mach. Reciprocally, the intersection of all $s \in \mathcal{S}$ having the property p is the only possible indication of the property. It then entails the same risks to admit the existence of $\mathcal{P}$ than the admission of the existence of $\mathcal{S}$, we must admit both or none of them, for otherwise we incur in arbitrariness. Mach restricts these considerations to regard observable objects, which is mere willingness and not a logical demand, for if it were true, we could never speak of the properties of mathematical objects such as vector spaces or numbers. In conclusion, the fact that the search for the essence in Mach's method fails does not mean that there is no essence, it only means what it shows: the search cannot reach the target, it is an inadequate searching method. Having failed to grasp the universal, Mach is then forced to pick one of the particulars (the "simplest one" in the words of Hertz) and to think in terms of it extending the results obtained to the whole class by invocation of analogy[89]. There is then

[88] A similar discussion is found in [Sartre, 1966, p. 3], see footnote 79.

[89] It is interesting to compare the present view with Mach's position in [Mach, 2012, The economical nature of physical inquiry]. We read:

> Nature exists once only. Our schematic mental imitation alone produces like events. Only in the mind, therefore, does the mutual dependence of certain features exist.

no abolishment of metaphysics but rather what is abolished are higher levels of abstraction at the cost of introducing subjectiveness.

6.4.2.2. *The mediation principle and the dialectical openings to understanding.*

Dialectical openings. Our discussion of the construction of science is based upon the repeated action of synthetic cognition [Hegel, 2001a, §1720].

This consideration takes as given the multiplicity of determinations, the observable. It then underplays the fact that in order to perceive as multiple what can be clearly argued that is different, we need to disregard some part of what is determined in each element of the multiple. The creation of concepts that do not refer to previous understanding but rather are inter-defined (usually, just two concepts indicating the ideal and opposing extremes of a perceived difference), is called a *dialectical opening* (cf. Chapter 1). Abstraction, the grasping of something as a particular case of an universal ideal and the synthesis of the particulars in universals is a dialectical opening. However, not all openings are abstractions. The opening that recognises an Ego and an external world (SW) is more complex. When we recognise ourselves and at the same time that "there is something out there" whose action reaches us through the external senses, we become conscious of SW and create a dual of it, IW, our ideas of the world which lie in our consciousness alongside of the idea of self, this is, self-consciousness. At this point our road begins to depart from Hegel[90], as he addressed not IW but rather self-consciousness. The basic dialectical opening of natural science corresponds to the creation of the duality (SW, IW). However, the most striking form of dialectical opening is the one performed by the child conceiving space and time (see Section 6.4.1.2 and Chapter 1). This operation requires that there is a something (name it relative position with respect to ego, or position in space) that is not permanent, that

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> Let us endeavor now to summarise the results of our survey. In the economical schematism of science lie both its strength and its weakness. Facts are always represented at a sacrifice of completeness and never with greater precision than fits the needs of the moment. The incongruence between thought and experience, therefore, will continue to subsist as long as the two pursue their course side by side; but it will be continually diminished. In reality, the point involved is always the completion of some partial experience; the derivation of one portion of a phenomenon from some other. In this act our ideas must be based directly upon sensations. We call this measuring.

In Mach "schematic mental imitation" corresponds to the role of abstraction, now exercised by blurred images. Rather than decorative detail or particularities, what is sacrificed –according to Mach– is completeness. This action in Mach parallels the action of Π. The restitution of the particulars that corresponds to Γ becomes in Mach "the derivation of one portion of a phenomenon from some other", which requires analogy and imagination. Thus, the central difference is Mach's relying on "imitation" as opposed to our relying on abstraction. Mach [2012, The principle of comparison in physics] explains:

> The adoption of a theory, however, always involves a danger. For a theory puts in the place of a fact A in thought, always a different, but simpler and more familiar fact B, which in some relations can mentally represent A, but for the very reason that it is different, in other relations cannot represent it. If now, as may readily happen, sufficient care is not exercised, the most fruitful theory may, in special circumstances, become a downright obstacle to inquiry.

Mach names the described substitution as "analogy".

[90] Hegel addresses self-consciousness in his first book, The phenomenology of spirit.

is a non-permanent property of the object. Further, if there are non-permanent matters, the idea of non-permanent becomes the idea of change, later *time*. Any attempt to explain one element in the dualities ego-alter, permanence-change calls for the other term, any attempt to explain position requires an idea of time, and an idea of object. Thus, the elements of the descriptions are interdependent, they have been constructed by idealising (taking to extremes) perceived differences and all of them together open the possibility of organising what reaches our senses. A dialectical opening institutes the terms that make possible to organise the idealised world and with it the Sensorial World, it makes understanding possible.

Notice that dialectical openings operate on the basis of perceived differences which are ideated into complementary options within their universe of application. Being complementary, both of them are the Universe minus its complement: the negation of each other. We emphasise that the differences are perceived, they belong to SW. Notice further, that the frequently encountered duality essence-appearance is not a dialectic opening when the essence is thought of as the *noumen*. In such a case, it leaves all the perceived as appearance, as opposed to a metaphysical entity: the essence, the "thing-in-itself". In this sense, dialectical openings realise the elimination of metaphysics as the Vienna Circle sought, but preserve the abstract with a necessary nexus to the observable. But if we say: "the essence of a key is to open doors" we are constructing a valid duality where essence corresponds to "function" (relation with other objects) and we are leaving its material form in the appearance.

Mediation principle. We do not usually accept as reasonable that which appears out of nothingness as self-evident assertions. We normally request a new rational belief to be derived (mediated) by acceptable argumentation from accepted beliefs. This recurrent form of reasoning cannot be pursued indefinitely. It comes to an end when we reach a point in which beliefs can no longer be derived from other accepted beliefs. At this point there seems to be only one option: Either we make explicit a layer of arbitrary assumptions (axioms) which is the opaque end that reason lets us see, or we find a set of opposing concepts and ideas that in their interplay constitute the foundation of our discussion; the dialectical openings.

Axiomatisation turns natural science into exact science, physics into mathematics, by removing the links between IW and SW. However, a purely abstract science is void. Instead of pushing physics into the exact sciences we must consider mathematics as a natural science, being the fundamental elements of mathematics the idealisation of quantitative relations in the observable world, which are always in relation to qualities [Usó-Doménech et al., 2022]. Thus, projecting out the quality in SW we obtain the quantity. The operation requests us to conceive the dialectical opening quality-quantity along which we make the projection.

6.4.2.3. *Cognitive surpass.* The introduction of explanatory hypothesis, the process of abduction, is subject to the control of rationality and to the condition that the newly introduced hypothesis explains a class of problems larger than the one that motivated it, this is, that the hypothesis bears some of the main ingredients of *cognitive surpass* [Piaget and García, 1982] and offers itself more openly to refutation. However, the requisites for the acceptance of explanatory hypotheses (i.e., to be able to stand in front of refutation attempts) say little about the method of production.

6.4.2.4. *The continuity principle (reduction to the obvious/evident).* Argumentations are constructed in such a way that they rest upon small units we consider evident or obvious. Yet, what is obvious or evident for some, may not be so for others. One of the forms in which we usually identify potentially irrational arguments is by detecting hiatus or *lacunae* in the argumentation. The request "please fill in the gap" quite often reveals a belief that cannot be supported, while being necessary for the argument. On the contrary, the rational argumentation proceeds to fill the gaps by explaining how they consist of the concatenation of smaller pieces, iterating the process until the pieces are accepted as evident or obvious. This self-similar form corresponds to what in mathematics is called *continuity.*

6.4.2.5. *Logical action in front of contradictions.* Whenever a chain of deductive reasoning arrives to a contradiction, the whole chain is rejected. When the contradiction results from comparing theoretical prediction and experimental reality we speak of experimental refutation. The logical scheme can be depicted as $A \Rightarrow$ [consequences] $\Rightarrow B$ and B evaluates to $False$. No matter how pleasant the intermediate consequences are, there is no support for them. The most evident example is the hypothesis of the ether which is fundamental for the proposition of Maxwell's displacement field. Discarding the existence of the ether (following empirical evidence) under the present principle would mean the refutation of the hypothesis as well as its consequences. Yet, in general $A = a_1 \& \ldots \& a_n$, i.e., A may be a composite statement consisting of different parts. Only the hypothesis and consequences involved in the deduction of B are necessarily affected by the falsity of B. Thus, part of the theory survives and only part needs to be reconstructed under new hypotheses.

There is another instance of the same logical scheme which is not usually considered, namely when the contradiction stems from the logical structure of the theory (e.g., inconsistent postulates). Assume A is True, then $\sim A$ is False. The construction $A \Rightarrow$ [consequences] $\Rightarrow (\sim A)$ discloses an internal contradiction of the theory and, as above, it forces us to reject the full chain.

Again, it is worth to realise that a refuted theory may require only some minor repair since usually A is a composite statement. It is enough for one of its terms to be wrong for the theory to be refuted. Refutation does not mean "throw away all your thoughts".

6.4.2.6. *Example: demarcation of a non-scientific belief.* Let us show how the requisites proposed in this Section change our perception of what is acceptable as scientific and what is not. Suppose we have a belief, T such as "All swans are white", and we have a form of determining what a swan is without considering its colour, call it A; we have that $T\&A \Rightarrow A\&W$, meaning that if we believe the theory it "explains" that the swan I am observing is white. The most immediate reason why such a belief is to be called non scientific is that has not been explicitly linked to observations, thus, we cannot recognise the construction of a theory, which is, at the end, what is to be subject to appraisal.

Let us now give reasons for our belief: we have been observing fowl during ten years at a lake nearby our home. In these years of observations we have recognised N swans (mostly by their neck, beak and swimming, say) and the totality of them were white. Call these observations O and we have conjectured T out of O, this is our basic theory. Would now our belief be scientific (given the fundaments)? Our observation consists of a triple: a statistics, a place and a time frame. The theory T is produced by projecting out place and time frame. In order to confront the theory with new data we have to produce statistics in other places and time frames. Let us check the requirements of rationality. The initial violation of the mediation principle has been repaired in our second attempt. Is there continuity? What allows us to go from the observed into the idealised theory? As we shall presently see, the answer is: a rudimentary version of probability theory.

Actually, we have no support to disregard the alternative explanation that the observed fact occurs only in the lake we observe or in some lakes or period of time, all doubts that tell us that our theory is not under a firm ground. We may restrict the theory to our lake and the observed period of time. During that period other bird-watchers might have collected data as well. In this case, statistics would allow no more than establishing a bound, p^*, on the probability, p, of detecting an individual not being white in the experimental situation we are exploring. Let $P(p; N)$ be the probability of observing non-white swans in N trials. Hence, for a null observation record, given P we can estimate p^* as the solution of the equation $P(p^*, N) = \frac{1}{2}$. For all $p < p^*$ it is more likely not to observe a non-white swan in N trials than observing it. Hence, the observations only allow us to say that having *not* observed a non-white swan is not just bad luck but rather what it is expected for such a probability p. But then, why have we chosen $p = 0$ in our theory T? We meet arbitrariness once again. With the sole support of statistics we cannot make such a bold theory. We would have to make then a third version of the theory which changes our belief for something in terms of probabilities, which eventually will cast doubts about the quality of our statistics: Do we have bias in our sampling?

The example shows how we proceed from an initial belief, a hunch, critically searching for its fundaments using some of the rules for reasoning that are not (at least in this presentation) formal logic. The lesson gained from the observation of white swans in some place and period of time is: If things do not change too much since I made my observations, I am likely to encounter only white swans at that place next time. This suggests ways to improve the theory through further enquiry: I might suspect genetic similarities among the swans and also its counterpart: That there may be more genetic diversity in swans at other locations (or different epochs of time, but changing location is feasible while moving to "another time" is not). The "hunch" and its criticism trigger a research programme by putting together other knowledge and the awareness of the limitations of my observations. This generates an ampliative hypothesis to investigate.

6.4.2.7. *Example: the principle of relativity.* Most physicists, including us, accept the principle of relativity as correct. Why is it so? While Einstein does not offer any argument [Einstein, 1905a], Poincaré does [Poincaré, 1913b, Ch. VII], when he states:

> The movement of any system whatever ought to obey the same laws, whether it is referred to fixed axes or to the movable axes which are implied in uniform motion in a straight line. This is the principle of relative motion; it is imposed upon us for two reasons: the commonest experiment confirms it; the consideration of the contrary hypothesis is singularly repugnant to the mind.

Poincaré comes to no better argument after several pages dwelling in Newton's mechanics as it was taught at his time (based upon absolute space). No search in the realm of mechanics will serve the purpose of finding the foundations of the principle after Newton's axioms. The foundations are to be found behind them, a matter Newton did not discuss at large. Moreover, the little he did discuss was misunderstood, such as the notion of "True motion" which was shadowed and finally replaced by the notion of "motion in absolute space" which is not in Newton (see Sections 1.2.2 and 1.A.5).

The idea of the relativity principle as proposed by Poincaré and others under various names (such as "symmetry principle" [Mach, 1919]) is a belief coming from the habit instructed to physicists by the teaching of Newton's mechanics. Observational inferences such as "All swans are white" can be put to experimental test (and in this case proved wrong by e.g., displaying black swans from Australia or black-necked swans from South America). On the contrary, the relativity principle is not observational. To assess its truth value we must seek its foundation in the demands of reasoning. The principle can be regarded as a special case of the no arbitrariness principle (NAP) since it states that any choice of reference frame within the class of "admissible systems" is as good as any other. To pick a system is an arbitrary decision, then arbitrary decisions should be transparent for the laws of physics. For Newton's laws of dynamics this amounts to the Galilean group of transformations that relate positions and velocities as recorded by any (inertial) system without altering the physical laws (Newton's equations). The transformations must form a group since the composition of valid transformations is required to be another valid transformation.

At those times, the issue was how to merge electrodynamics with the principle of relativity. Maxwell's equations were known to obey a symmetry with respect to Lorentz transformations. Hence, it appeared as Lorentz transformations were the needed connection between admissible systems. It is interesting to examine the argument in [Einstein, 1940]:

> The so-called special or restricted relativity theory is based on the fact that Maxwell's equations (and thus the law of propagation of light in empty space) are converted in equations of the same form when they undergo Lorentz transformations. This formal property of the Maxwell equations is supplemented by our fairly secure empirical knowledge that the laws of physics are the same with respect to all inertial

> systems. This leads to the result that the Lorentz transformations –applied to space and time coordinates– must govern the transition from one inertial system to any other.

In the first place, the conclusion is not deducible from the premises. Secondly, the Lorentz transformations (LT) do not form a group, since the associative law fails[91]. The Lie algebra associated to the group of Poincaré-Lorentz symmetries has dimension six and what is needed has dimension three[92].

Third, there is nothing such as "...empirical knowledge that the laws of physics are the same with respect to all inertial systems". Inertial systems, as well as the laws of physics, are abstract concepts. The invariance of natural laws with respect to the choice of inertial system is a matter of reason and of habitus. Einstein refers to the habitus developed when studying Newtonian mechanics where the concept of "inertial system" was originated. What belongs to the habitus and what to the empirical knowledge was in any case clearer for Poincaré than for Einstein.

The student in physics and the learned physicist know about Galilean transformations in the context of Newton's laws. They can legitimately ask: why this symmetry must be the symmetry of space instead of some other set of symmetries? What textbooks fail to indicate in this respect is that Newton aimed at describing relative motion, an aim that the facilitation of Newtonian physics forgot. Relative motion relates to the *SW* in a direct form, it is real in the terms presented here. If, in order to facilitate mathematical expressions, we introduce a Cartesian frame (space), the reference point of the frame can be changed almost arbitrarily with time (provided we satisfy proper continuity conditions) and this is done without altering the relative motion (let us call this group the *inertial group*). It is this set of transformations what matters to the representation of the observable-real. A subgroup of this group corresponds to Galilean transformations, that preserve as well true motion and hence the form of Newton's equations. Thus, any transformation aiming at replacing Galilean transformations must be a subgroup of the inertial group, or otherwise it must present the form in which the new space relates to the observable one, making room for the replacement transformations that are to be proposed. Educated habits play a fundamental role in this matter.

We teach in physics to represent motion by a graph, this is, we create a pedagogical space (not a movie) called space-time: $\mathbb{R}^{1+1}$. This opens the possibility of mistaking time with a second spatial coordinate. Indeed, if the student is asked to represent a trajectory of constant speed going from (x_0, t_0) to (x_1, t_1) they will efficiently draw a straight line. However, if we ask: which was the distance

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[91] This observation already indicates that LT's cannot connect different inertial systems, since such connections are automorphisms of the inertial systems and as such must form a group (this is today an elementary theorem in Category theory).

[92] A defender of relativity would argue that the three group generators not considered correspond to rotations. This is true in abstract terms, but the necessary correspondence with rotations in the Sensorial World is not present. The rotations correspond to $L(-(u \oplus v)) \circ (L(u) \circ L(v))$ [Gilmore, 1974, p. 503], where $\oplus$ stands for Einstein's addition of velocities and $L(u)$ is a Lorentz transformation based on the velocity u. To imagine in our minds the consecutive application of three Lorentz transformations does not enact the rotation of physical objects. There is no correspondence between actually rotating a given object and the rotation group regarded as a subgroup of the Poincaré-Lorentz group.

travelled by the represented body? they might answer $d = |x_1 - x_0|$ more likely than coming to realise that there is not enough information to answer the question. Notice that in any other representation (y, t) with $y = x + z(t)$ the answer would become $d' = |x_1 - x_0 + z(t_1) - z(t_0)|$ depending on the arbitrary election of frame used for the representation (d actually corresponds to the change in the relative position between body and origin of coordinates, it is not a property of the body). The information in space-time is not enough to reconstruct distances and relative positions. Space-time is subjective, it depends on the observer, it is not directly linked to SW but rather it is the result of intuitions cemented by the teaching of physics.

Being such the case, there is a possibility for other rules of reasoning being broken by special relativity. The exploration points to the meaning of the velocity in a Lorentz transformation. First, the meaning of velocities in the context of Electromagnetic theory changed from "relative velocities" in experiments, to velocities relative to the ether later and finally to velocities with respect to a reference frame. Such changes are a signal of trouble in itself [Assis, 1994]. Being such the case, there is a possibility for other rules of reasoning being broken by special relativity. The exploration points to the meaning of the velocity in a Lorentz transformationas it has been discussed in Section 2.A and Section 5.1. This matter puts us in front of the dilemma: either special relativity is imperfect or we have to abandon the hopes for science to be based only in observations and reasoning. Historically, the second alternative was taken.

Indeed, Einstein advocates that there is no abstraction (or any other relation) between observations and the theories at the time of production [Einstein, 1936], they arise by free invention and are validated by their results. The failure to recognise this problem originates in the suppression of abstraction:

> An adherent to the theory of abstraction or induction might call our layers "degrees of abstraction"; but, I do not consider it justifiable to veil the logical independence of the concept from the sense experiences. The relation is not analogous to that of soup to beef but rather of wardrobe number to overcoat. [Einstein, 1936]

Einstein gives "logical independence" to the concept from its conception originated in our sensorial experiences. Adopting "free invention" implies to break the connections with nature, to have purely abstract concepts, detached from their conceptualisation. Science would then no longer be a matter of understanding nature, but rather a sort of game. In Einstein [1936] words:

> It is an outcome of faith that nature –as she is perceptible to our five senses– takes the character of such a well formulated puzzle.

6.4.3. The "marvellous self-correcting property of Reason". Laudan [1981, p. 188] has stated that

> No one was able to suggest plausible rules for modifying earlier theories in the face of new evidence so as to produce demonstrably superior replacements

in support of his general idea that Peirce's "Self-corrective thesis" [Laudan, 1981, Chapter 14] cannot be demonstrated. He claims further that [Laudan, 1981, p. 239]:

> Peirce, in short, gives no persuasive arguments to establish that qualitative induction is either strongly or weakly self-corrective. [...] What the facts do not show, of course, is how the hypothesis is to be altered so as to bring it closer to the truth.

Laudan's criticism to Peirce is rather a criticism to Laudan's own interpretation of Peirce. Indeed, Peirce writes

> So it appears that **this marvellous self-correcting property of Reason**, which Hegel made so much of, **belongs to every sort of science**, although it appears as essential, intrinsic, and inevitable only in the highest type of reasoning, which is induction. [Peirce, 1994, CP 5.579] [the part quoted by Laudan is highlighted, and Laudan's sentence finishes with: 'and every branch of scientific inquiry exhibits "the vital power of self-correction".[Peirce, 1994, CP 5.582]']

Laudan's selective quotation left behind a number of fundamental ingredients, criticising thereafter a weaker version of self-correction rather than Peirce's developed concept. It is not clear whether Laudan restricts science to the science recognised as such by Peirce [CP 1.43–1.45] or not.

The rules for self correction are not to be sought outside us, as Laudan appears to demand, but inside us, they are the tutors of our reasoning. Improving our theories and our reasoning requires the use of critical reason. Recalling that deduction is mechanistic, it follows logic and mathematics and goes from the universal towards the particular, and that critical thinking moves backwards towards the foundations, from the particular towards the universal, we can grasp that critical thinking cannot be mechanistic. In some sense it is a sort of inverse process to logic. In logic and mathematics the rules of transformation are known, so the elaboration of the particular case results of a somewhat mechanical application of these rules. But in retrogression what is certain is the observed, the particular cases and what have to be produced are the rules. Laudan's criticism parallels the problems that several well respected logicians experimented with dialectics (see Subsection 1.B.2).

The connection with *all* reason, with logic [CP 2.191] and with Hegel are missing, the latter being important to grasp abstraction, cognitive surpass and the grounding of hypotheses (see 6.2). Further, Laudan focuses his criticism in a version of inductive reasoning isolated from other reasoning processes, hence disregarding the "vital power" of reason as a unity. In all, the possibility of learning from our mistakes and improving on them with the aid of reason is degraded, if not neglected.

The self-correcting property belongs to reason and it is inherited by science only as much as science inherits reason. In this Section we highlight the existence of "rules of reasoning" restricting science in the same form in which Peirce did. Along this work, we have shown how the concept of abstraction can be made a little bit more abstract (Section 6.3), i.e., offering a closer focus on the essence/nature of the concept, and how this abstraction immediately opens the possibility of showing e.g., that Mach's relationism is actually a secular form of absolute space. We have improved the "Relativity principle" by showing that in terms of clear thinking it is not a separate principle but a particular application of the, more demanding, No arbitrariness principle. We have further shown how one classical example of Inference to the best explanation can be polished by making it rational according to the given rules (Section 6.4.2.6).

In short, when the production of true beliefs as a process, science, and logic are regarded –inspired in Peirce's ideas– as particular expressions of reason following a few identified rules, the "self-correcting property of Reason" can hardly be denied.

6.4.4. Multiple abstract projections and the case of science. The idealisation or abstraction of an observed phenomena is performed with the aim of organising our view of it, linking the new facts to pre-existing matters in our understanding. Such operation is motivated by the need of answering questions regarding the phenomena. Thus, both the questions being posed and the pre-existing knowledge suggest which features of the phenomena carry the potential for *explanation* (providing an answer to the questions) and which do not. This process is followed by logical elaboration and interpretation that provides the opportunity for contrasting the ideal with the observed, and, in case of refutation it triggers a new attempt at producing understanding. The process is directed by the posed questions, and as such, different idealisations are possible for different sets of questions. The observable real is then crossed by several idealisations, each one can be said to correspond to a *dimension* of the phenomena. The associated projections on IW characterising these aspects are in principle considered independent of each other, something that will later be modified by synthetic judgement, that confronts the alleged independence. Let us illustrate this process with the idea of "science as it is practised".

The notions of science so far discussed correspond to an ideal, flawless functioning, science. The practice of science develops in a society which is part of a civilisational movement, thus there is a science idealised in terms of its relation to the society at large. Science is practised by human beings that constitute a particular field of symbolic production [Bourdieu, 1999]. Thus, we easily find three different dimensions in the consideration of science.

From the point of view of society in general the goals of science are often related to the production of goods and practices that enhance well being. Central to well being is technoscience, geared towards the production of new goods, enhancing comfort and capabilities. Technoscience frequently adopts the criteria proper of technology and focuses on predictive success. The quality of this science is hence rooted in prediction. If something *works*, this is taken as support for it being *correct*. The foundation of scientific theories is subordinated to their success capabilities. In the schema of Figure 6.4.1, the focus is on Γ.

In addition, science is requested to guide some important decisions. For example, decisions in matters of global warming, epidemics, nuclear energy safety, human environmental impact and the extinction of species. In such endeavours, the contrastive comparison of the predictions is not possible. This aspect has been called "science in the post-normal age" [Funtowicz and Ravetz, 1993, Waltner-Toews et al., 2020]. Such practice is forced to root its quality in the elements Π, ϕ of Figure 6.4.1, since Γ is not available.

The most traditional perspective is that of science as the search for harmony and understanding. All three elements Π, ϕ, Γ are then equally important and they cooperate (e.g., via autocorrection) to enhance understanding.

Thus, differences in epistemology are to be expected in correspondence with the demand put on science by society at large.

This situation is a constant source of misunderstanding. It may be argued that the "correct" relation between theory and observation is the one reflecting the current practices of the community of scientists, as if the practices of scientists were not conditioned by the necessity of justifying science in front of the supporting society (society at large, governments, granting institutions, etc.) or were not conditioned by the need to conform to established practices of their scientific social field. There have been attempts at explaining science as a practice directed towards the acquisition of knowledge in terms of features of the social structure of the field, such as competition for resources and social respect. A third source of misunderstanding is to believe that what has been observed for science in some age (say after World War II) can be used to explain the development of science in another age (e.g., before the second industrial revolution). To consider science as "that which is analogous to what is currently observable" is to operate against the process of abstraction, which, as Piaget taught us, is geared toward the discovery of the possible as opposite to the given. It should then be considered a political act of conservatism that deprive us of ideals and the exercising thoughts directed towards the search of the foundations: the critical thinking.

6.5. Final thoughts

The idea in Peirce that it is possible to agree about the real depends on the criteria for the cessation of doubt and the admissibility criteria for hypotheses. We have shown that there are different criteria in use for both actions and as a consequence we are not reaching truth but rather opinion. If (the pragmaticist's) Truth is to be achieved, clear thinking must take the ultimate word above the turbid thinking that leads to opinion. The restoration of critical thinking requires the rejection of arbitrariness and recovering a rationality that goes beyond instrumental reason. The arc of the construction leads then to investigate reason in its objective form rescuing the role of abstraction, indicating it as a prerequisite for rational abduction.

In Section 6.3.3 we discuss how spatial relations link to a deeper abstraction level than the traditional notion of space, by discussing Ernst Mach as a representative of empiricism. In several respects, Mach is simply unable to understand Newton and/or the relational space; he needs a first reference of position and motion, and he boldly sustains the "fixed stars" as such reference when confronted with more abstract ideas such as those of Lange. However, space is only a production of the child, made out of spatial relations and the suppression of her/himself from the total picture. In an early stage of development we fail to perceive us as one arbitrary particular of a universal, we fail to produce abstraction. Spatial relations are real, their observable form corresponds to relative positions of objects, space is only a convenient form of representing them. The idea that space does not exist as such, which is clear in Kant, in Poincaré and later in Piaget, simply obfuscates a great mind such as a young (30 years old) Bertrand Russell. Actually, the idea that the form in which we ordered the observable in our early infancy might not be the universal form for understanding appears as inconceivable to some minds. To sustain physics on the phantasy of a space that is no longer linked to completely observable events, rather than on the observable spatial relations, "simply" requires more phantasy: things that are unobservable except for their alleged consequences, the observable facts. We have already been warned by Faraday that what began by being a conjecture too often becomes a belief just by habituation. His lesson is of such relevance that we indulge in repeating it (cf. Section 2.B):

> But it is always safe and philosophic to distinguish, as much as is in our power, fact from theory; the experience of past ages is sufficient to show us the wisdom of such a course; and considering the constant tendency of the mind to rest on an assumption, and, when it answers every present purpose, to forget that it is an assumption, we ought to remember that it, in such cases, becomes a prejudice, and inevitably interferes, more or less, with a clear-sighted judgment. [Faraday, 1844, p. 285]

Is it possible for a science based upon reason and experience to recover the lost track? How is rational retroduction different from other forms of inference? We discussed some characteristics of rational abduction/retroduction in Section 6.4. We have made an effort to produce a few rules of rational thinking, and showed how they help to construct more solid beliefs. We hope it is just the beginning of a collective task long overdue, and that other scientists and philosophers will contribute their own rules. In front of us rises the most formidable task: to rethink the possibilities of humanity and the life on Earth. To believe that the same science that gave us the menace of nuclear destruction, global warming and an accelerated extinction of species will give us the means to avoid catastrophe is only the characteristic insistence of the dogmatic. Critical thinking, a critical philosophical attitude and critical science are urgently needed alongside technoscience.

If we drop the requirement for explanations to be rational, we can then include elements of turbid thinking as explanations, for example, we may accept to stop doubting when we have an analogy. It is not the formal part of abduction –the logic in abduction– what changes but rather its quality. Once rationality has been removed from hypothesis making and from theory testing, we preclude clear thinking from taking us back into reason.

What we call rational retroduction is not merely a reasoning step, but rather the integrated process of generating abductive hypotheses grounded in reason and in previous knowledge while accepting them tentatively only after all efforts to refute them have failed. The rules of reasoning summarise the guidance given by reason when attempting to improve theories. We present a dialectical view of abstraction, contesting Mach's criticism. If making abstractions is a necessity for understanding, abstraction cannot be rejected.

CHAPTER 7

Closing

The present work, constructed much under the old epistemological view shows that physics depends on epistemology, and through epistemology it depends on social interests that have nothing to do with Nature. The immediate utilitarian consequences appear as unchanged but the foundations for advancing in the understanding of Nature appear as dubious if the instrumentalist perspective is adopted.

We believe we have shown by example that when we adopt a different *epistemic praxis* such as we have done in this work, we reach a different theoretical understanding, actually a deeper understanding when the epistemic praxis is more demanding, as it is the relation between our dualist phenomenology (see Chapter 2) and instrumentalism. Thus, physics depends not only on the observable natural phenomena but it depends as well on our philosophical disposition. Yet, if the concept of theory is weakened so as to be reduced to the equations (cf. [p. 21 Hertz, 1893]), the distinctions between theories mostly fade out, the phenomenological link disappears and interpretation crops up to guide the use of the formulae after sacrificing the unity of thought, critical and phenomenological actions.

In summary, there is no compelling reason to believe scientific results that have been achieved by weakening the old scientific demand for Truth into the current demand of usefulness. Newtonian physics is not incompatible with EM, as it is usually preached; and both of them can live in harmony with quantum mechanics. There is no reason for abandoning (relational) Cartesian geometry or to consider time as an "odd" spatial coordinate. There is no license to use formulae outside their range of validity as it is too often done.

In the course of this investigation we have encountered misrepresentations of the thoughts and writings of the natural philosophers that developed physics until the middle of the XIXth Century, when the *scientist* was born. Abridged representations of their ideas and writings creep in as soon as the creators fade out. We have already mentioned how absolute space is attributed to Newton's mechanics and "True motion" disappears (see Section 1.2.2), Faraday becomes a supporter of the ether rather than the careful philosopher he was, absolutely inclined to entertain doubts as much as possible avoiding to precipitate into simplifying inventions (see for example Section 2.B).

Maxwell's plead to consider the hypothesis of the ether as worth of research [p. 493 Maxwell, 1873] was transformed as well into a belief when the story was told. Yielding to the needs of the new social position of science (and its emerging epistemic praxis), Maxwell's theory was deprived of its mathematical construction lowering the acceptance standard from correct into plausible.

In our own experience, we have spent more time undoing promissory hunches than constructing correct reasoning. It is worth noticing that such hunches trigger exploratory actions and as such are important. It is believing, instead of doubting them, what makes them prejudice. What appears to us as correct is made of the debris of our errors.

The great minds that constructed special relativity and quantum theory were passengers of their epoch, a time when "conquering nature" (the old imperial dream of Francis Bacon) had finally taken prevalence over the "understanding nature," sought by his contemporary Galileo. Our own work cannot escape the rule, even though we are not "great minds". We live in a time where Nature reminds us that we must understand ki[93] and, consequently, love ki. Science must then rescue the teaching of the old masters, the natural philosophers, and their practice of critical thinking.

This book intends to reflect the complex structure of the knowledge of Nature. The exposition requires a determined sequence but the reader is encouraged to create their own sequence of readings, going back and forth in the text. There is some risk in so doing, the risk that the meaning we intend for a concept could be different from the meaning the reader has given to the words or concepts. The problems can be turned into advantages if the reader dares to explore what lurks behind the differences. We are persuaded that every error, every failure is an opportunity to learn. It is not a matter of trial and error as it is often put. Every error should trigger an inquiry on our beliefs, the deeper we dig into the causes of our errors, the more radical our new theories will be, the greater unity will be achieved and the higher the abstraction level that will be reached. Do not believe, doubt. Our ability of entertaining doubts, of letting doubts evolve until the contradictions become so familiar that reveal their clues, determine our possibilities of improving.

There are several themes in this book, in an analytical view we can identify the relations of natural science with: the sensory reality (say Nature), society and its ethos, psychological aspects that determine the cessation of doubt and the social structure of the scientific field.

We have emphasised physics as a science concerned with Nature, while quite often it is said to be an exact science. The latter expression refers to the fact that theories in physics, more often than not, are presented in mathematical, exact, forms. Exactness is not a property of our knowledge of Nature, thus physical theories are mathematical because of our construction of them through procedures such as idealisation. The relation between Nature and idealisations (physical theories, beliefs) is produced under the surveillance of the subject and involves psychological aspects both innate and developed in earlier periods of our life.

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[93] The word ki has been proposed as a new and respectful pronoun for Nature and all what is part of Nature. The word comes from Anishinaabe's language but relates as well to the Japanese and Chinese Ki and to the old English kin (like in kinship).

Our possibilities of adapting to the environment, a necessity for all living things, determines a conflict between what comes from within and what comes from outside. External influences come at least in two forms, social influences and reality. It may be argued that society is part of reality. However, the overwhelming influence of society in the life of human beings encourages us to make a distinction between what reaches us with and without the (determining) mediation of other humans. The extreme terms of the dialectic are then natural versus social.

Reasoning is our main adaptive tool with respect to Nature. As far as we know, Nature does not lie. Lying is a social action, it needs the involvement of at least two people. Intentionally lying is a social tool, useful to manipulate other people. It is perpetuated as a learned attitude in society, it consists in inducing others to believe what we do not believe. When we state that Nature does not lie, we mean that there is no evidence of any manipulative attempt from Nature toward us. Nature simply is, its Truth is encapsulated in what we can observe of it. Thus, the primary Truth is in the observations, the most direct and unperturbed, the better.

Direct observations are at hand in mechanics, we can indicate: "the leaf is falling" and after a few experiences we would have synthesised a notion of falling. A few cases more and we will reach the notion of moving. The idea that movement is relational may take some intervention of the master since clear and direct experience is not always available. If not taught by example, the idea of a relational space might not emerge early enough in our life and producing it might be a difficult task. We must be conscious that examples are low quality substitutes of experiences, the territory of examples is imagination, not real life. The problems with the relational space evidenced by Russell, Mach and Einstein among others, show that what is produced by teaching "does not enter into his [their] very being, but still remains alien to his [their] true nature" (as Humboldt said). Imagination, fantasy, is not a proper substitute for sensorial input, it is a weak substitute biased towards visual memories. In practice, fantasy apparently made it impossible to apprehend Newton's idea of "True motion", except for a few minds along the centuries. Resource to imagination, to examples, establishes the habit of contrasting ideas with the unreal imagined rather than with the real perceived. The abuse of imagination begins with the ether and since then it has never ceased.

When we advance in our exploration of Nature, for example into the electromagnetic realm, direct evidence becomes harder to access and indirect evidence is used. The quest to unify "Voltaic electricity" and "static electricity" was in part helped by the experience of electrical discharges of both types that felt alike on the researchers' bodies. However, most of the research was mediated by previous beliefs, some times correct, others sound and quite often incorrect. Understanding the new, generating new theories, then requires to modify previous theories, it requires to follow the source of conflict and error backwards, to retrogress. A science without retrogression is a science without critical thinking, a science that has left reason behind (hence irrational to some degree) since only reason has the self improving ability [Peirce, 1994, CP 5.579].

The unity of Science is then a demand of our way of learning, of our launching of new attempts at learning from the platform provided by our beliefs (theories). If new findings make our old beliefs tremble we have two forms of proceeding. We may retrogress into the causes of the trembling, improve our theories and preserve the unity of knowledge. Alternatively we could patch our old beliefs determining exclusion zones where they are conveniently substituted by local theories. The patching method relates to technology. When the theory is conceived as a tool and we find some difficulty, we simply replace the tool with one specialised in dealing with that difficulty. This method is in itself instrumentalist, it leads towards specialisation and narrows our empirical basis (the conflicting phenomenon is excluded from the empirical basis of the previous theories). In this form it makes retrogression even more difficult and by limiting the scope of the old theory makes its improvement impossible, thus becoming a dogma. In contrast, the demand of consilience, the demand for unity of knowledge is a demand of rationality.

Critical thinking is a problem for "the powers that be". Because of its universal character, reason cannot be confined to a realm, we cannot be reasonable in (say) biology and unreasonable in other matters, the same unity of knowledge built in reason forbids it. Further, since reasoning is fostered by doubting, a rational person might find interesting to doubt for example the "divine origin of the king", the "sovereignty of the people" in democratic regimes or think that "... it is impossible for the representative of a plurality to be so true an organ of all the opinions of the represented" [p. 30, Humboldt, 1792 Printed 1854]. Dogmatism is then a form of preserving the social ranks from the menace implied in perceiving its true origins, the reality of the domination–submission relations. Even political revolutions aim at changing the dogma while preserving dogmatism.

In this sense, the period of the Enlightenment must be regarded as an extraordinary period where social conditions allowed for greater freedom to philosophers and as a result a more meaningful relation with Nature commanded by reason. The Enlightenment period came to an end when the society determined that reason has not the right to examine every matter in conjunction with its specific knowledge. When the "Tribunal of Reason" was abolished, scientists were granted the liberty of not being subject to that tribunal; each discipline claimed to have its own reasons [Beiser, 2014]. It was the establishment of independent sciences (disciplining of science) and the emergence of a new elite conformed by the guardians of the disciplined knowledge –we mean the scientists– what returned the western civilisation to its historical organisational axis: power. The relevance of Truth was derogated as well and utility was proclaimed the supreme value. The need of professionals well acquainted with the novelties of scientific knowledge, and able to expand it, lead to the professionalisation of science, but the professional uses his brain to implement the instructed knowledge. Exaggerating not too much, they become organic computers that operate with instructed ideas in programmed ways. The problem of setting up the organic computers is facilitated by didactic transpositions that finally are reproduced as the truth by the new generations that learned in these forms. We have discussed long enough how the ideas of great thinkers were distorted in the form of didactic transpositions, as well as the contempt with which other reputed thinkers have battered Truth in favour of success.

As Ortega y Gasset indicates, the wise men disappeared within a generation and the scientist, of whom he says “the majority of scientists help the general advance of science while shut up in the narrow cell of their laboratory, like the bee in the cell of its hive, or the turnspit in its wheel” [The barbarism of ’specialisation’, Ortega y Gasset, 1930], emerged as the finished representative of the mass men. Ortega y Gasset warned us that:

> The day when a genuine philosophy reigns again in Europe – the only thing that can save her–, it will be realised that man is, whether he wants it or not, a being constitutively forced to seek a higher instance. If he succeeds in finding it himself, he is an excellent man; if not, he is a mass man and needs to receive it from the former. [Ortega y Gasset, 1930]

By the time when Ortega y Gasset (OyG) was writing “The revolt of the masses”, between the two great European wars of the XXth Century, the mass men were producing their most (self)celebrated scientific work: quantum mechanics. A true success from the instrumentalist point of view yet one of its most irrational products. The time for a return to reason, a return of philosophy was not ripe then and it appears not to be ripe today (2025). With this work we intended to show how deeply entrenched in our science is utilitarianism, and how utilitarianism socially reproduces itself in science. We do not expect the mass men to abandon their privileges, nor to recognise the existence of the Enlightened men (“excellent” men in OyG). The thoughts of Chuang Tzu (see Subsection 1.A.1) apply to this matter. The revolt of the mass men against their destiny, against their true nature as OyG puts it, will be finished not by other humans but by Gea and if she cannot do it, the Universe will. In the most optimist scenario (for humanity), some humans will survive. This book has been written for them, we do not expect the mass men to yield to reason if not forced by Nature.

The changes in the sciences environment produced by the industrial revolution resemble the case of the Darwin Moth [Majerus, 2009]. The new social conditions for the development of science, the massive education of scientists at universities, in good measure tailored for the needs of the blossoming industry, selected the personalities most apt to excel in the new environment. The propensity to doubting, retrogressing, and criticising of those leaning strongly towards reason makes them not the ideal subject for indoctrination. In contrast, those that can substitute direct examination of the sensorial real by consideration of what is in their imagination have an adaptive advantage in the “economy of thought and teaching”. Lowering the requirements for acceptance of theories from rational to plausible promotes the adoption of analogy as a criteria for the cessation of doubt and facilitates the adoption of taught theories (the programming of the biological computer).

Logic tended to occupy all the space formerly associated to reason. However, the use of analogy in physics betrays logic. If A implies B and B is experimentally found to hold, logic warns us that A is not necessarily true. Still, agreement with experiment is often regarded as validation or "proof". The epistemic practice associated to "validation"[94] has been ported by physicists to other areas such as mathematical models of biological phenomena. This action imports a great risk as it implies the reversal of the burden of proof. Once a model has been "validated" (say tested in some particular case on some particular roughly measurable and computable prediction –or postdiction–) it is considered a valid model, a model (minor belief) that can be trusted until empirical evidence challenges it in a form in which it cannot longer be patched. In this form, the biological model can be constructed ignoring most of the biology and disastrous decisions can be taken under the counselling of "science".

The cessation of doubt by analogy conducts to conflicts. When the idea of a material ether where the electromagnetic action propagates became problematic its material character was abandoned and space became the site of the propagation. Einstein even argued that absolute space, which he associated with what was called "Newton's mechanics" at his time, was an abstract ether of the same form, and actually, he was correct. But the fantasy of absolute space was not enough to explain the photoelectric effect, an experience easier to explain with the propagation of the electromagnetic action by quasi particles carrying energy quanta. Conciliating both analogies became an impossible task. The unavoidable conflict between the metaphysical objects (photons and space-time) has been created and it requires nothing less than the abolition of mathematics to "resolve" it. The idea of point particles was luckier. Being first an idealisation, it became true without further examination continuing the habits of classical mechanics. Schrödinger, who challenged the idea, was simply ignored, receiving the social treatment reserved to all of those that challenge the guild of physics.

Classical mechanics, electromagnetism and quantum mechanics are all in conflict as it has been widely recognised. The conflict is so acute in quantum mechanics that a myriad of attempts to achieve consistency led to ludicrous dogmas (called interpretations) and a doctrine regarding measurements in quantum systems. In the same form that acceptance of the Trinity is central to most Christian cults, acceptance of paradoxes such as Schrödinger's cat, or the twin paradox became central to physics. We believe we have shown that nothing of this nonsense is needed and that, if we drop fantasy, we can achieve the unity of the three mentioned branches of physics.

[94]proof that something is correct: Oxford dictionary (consulted March 31, 2025).

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Index

Why is absolute space associated with Newton in the narrative of physics? What happened to "True motion", the central concept in Newton? Why do textbooks insist on “paradoxes" occurring in physics, forcing students to accept contradictions? Does the value of quantum mechanics lie solely in supplying a consistent mathematical scheme that would allow physicists to calculate measurable quantities, as Dirac claimed? Are physics equations the result of "free thinking" and a posteriori interpretation? When and why was critical reason left behind? For Galileo "Philosophy is written in this very great book that is continually open before our eyes (I say the universe)". When and why was the book closed? How did such a tragedy happen?

How do physics theories look like when we recover a complete reason?

We continue the work of philosophers (experimental and mathematical) reconstructing physics in full compliance with reason. We recover the unabridged Newton's mechanics, and complete Gauss' relational electromagnetism and Schrödinger's undulatory quantum theory, unifying all of them in one formulation and connecting with psychological and philosophical works and with the historical context.

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